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EFFECT OF CAVITIES ON NEUTRON DIFFUSION

by

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I. INTRODUCTION

The study of the effect of air gaps and cavities on neutron diffusion is of great practical importance in reactor physics because most research reactors contain air gaps and cavities of various forms such as beam holes, radiation cavities, and other experimental ports. These gaps and cavities result in a decrease in the average density of scattering atoms, and consequently the closely related diffusing properties change accordingly. In addition, a significant change in the localized neutron flux within and near these cavities is to be expected. It is generally known that the neutron flux distribution depends not only upon the diffusing properties and the geometry of the medium in the surroundings but also depends upon the shape, dimensions and location of the cavity introduced. In the enclosed cavity, however, where neutrons do not diffuse, the neutron flux is essentially determined by the streaming of neutrons from one position to another.

In many practical problems, it is desirable to know a qualitative relationship between the localized flux variation and other significant parameters when a cavity is introduced in the otherwise homogeneous medium. It is the purpose of this thesis to present the results of a theoretical investigation of the effect of cavities of various geometries on neutron diffusion. Analytical expressions have been developed for a non-multiplying and weakly absorbing medium in the presence of an isolated cylindrical and spherical cavity. Because of the complexity involved in formulating the condition at the wall of the cavity, the theoretical treatment has been confined to few simplified cases of practical interest.

The following physical models were chosen for the study:

I. Neutron flux distribution in a medium containing a cylindrical cavity.

   A. A cylindrical cavity in an infinite diffusing medium.

   B. A finite cylinder with a central cavity.

   C. A limiting case for a large cavity.

II. Neutron flux distribution in an infinite medium with a spherical cavity.

In order to develop analytical solutions for different cases, it is necessary to formulate a general condition at the surface of the cavity and this condition, in turn, depends on the behavior of neutrons at and within the boundary of the cavity. In the beginning of the investigation, a general relation that describes the behavior of neutrons in the absence of scattering and absorbing processes will be discussed. The behavior of neutrons is seen to satisfy the continuity equation in the absence of scattering atoms. This continuity equation, in turn, can be further reduced to a Laplace equation if the dimensions of the cavity are small compared with those of the surrounding medium and the scattering is isotropic at the boundary of the cavity.

Next, a fundamental equation of streaming neutrons will be developed on the basis of the diffusion theory. It will be shown that the applicability of the diffusion theory is restricted and the fundamental equation breaks down near the end of the channel. With conditions specified on and within the boundary of the cavity, it is possible to obtain analytical solutions for cases given above. Solutions will be obtained in the system with one-velocity thermal neutrons. All relationships are equally applicable for fast neutrons when considered as an one-velocity group.

Experimental evidence has shown that the effect of cavities on fast neutrons is much more significant than it is on thermal neutrons. It is, indeed, the case as the theoretical results indicate. In many cases, it is also desirable to know the corresponding slowing down density of fast neutrons in the system. A general relationship between the neutron flux and the slowing down density will also be given. In fact, the slowing down density is the Laplace transform of the neutron flux with respect to k^2, where k^2 is the square of the reciprocal of the diffusion length. Hence, it is always possible to obtain the slowing down density of a system once the flux is specified throughout the system.

II. REVIEW OF LITERATURE

The presence of air gaps and cavities in an otherwise homogeneous medium is known to introduce some changes of the nuclear properties of the system. Many articles and reports on this subject are available in the literature. Owing to the practical significance and the mathematical simplification, almost exclusively all the work that has been done has been based on the validity of the diffusion theory and the consideration of a cylindrical cavity with axial symmetry.

The qualitative discussion of the effect of cavities on a reacting medium was first given by Boherens in 1949 (3). He noted that the scattering atoms of the reactor are less closely spaced than they would be without holes. This effectively raises the mean free path of neutrons in the system and the average direct distance "as the crow flies" covered in the course of N collisions increases in the same proportion. In general, this increase of neutron paths must be proportional to the dimension of the hole through which they pass. If ϕ is the volume ratio of the holes to the material of the reactor and $J\phi$ is the volume of each hole, the amount by which each path is lengthened must be proportional to the quantity $\phi J/S$, where S is the surface area of the hole. In Boherens' paper, nothing was said on the localized change in the neutron flux along the surface of the hole.

In many practical problems, one is more interested in the localized variation of the neutron flux distribution than in the average values of nuclear parameters. The problem of determining the neutron flux along the surface of the hole involves a great mathematical problem. This problem has been discussed by several authors based on various assumptions and physical geometries.

The theoretical treatment of the nuclear reactor with a transverse gap was carried out by Chernick and Kaplan (6). In their work, the expressions of the neutron current along a cylindrical channel were given on the basis of both the diffusion theory and the transport theory. By comparing the diffusion theory and the transport theory treatments, it is suggested that the difference in results is due to the fact that the diffusion theory does not adequately represent the flux distribution in the neighborhood of the gap. Expressions are also given for the critical size of the reactor in the presence of a cavity. Similar discussions have also been given by many other authors on the effect of a large central void in cylindrical reactors. All these discussions are based on the assumptions that the diffusion theory is valid everywhere in the medium including the boundary, and no neutrons are created or absorbed in the void. Hence, the void acts essentially as a vacuum region surrounded by the homogeneous medium. Furthermore, the neutron current at the boundary of the void must satisfy a certain streaming condition. This condition, however, is not easy to deal with because the neutron flux at any point depends on the flux at every other point on the surface of the void.

Among all the works in which this streaming condition is formulated, perhaps the discussion given by Stummel (16) is most complete and general. He gave the basic relationships of the streaming neutrons on the boundary of the void based on a rigorous diffusion theory approach. By assuming the separability of the neutron flux in the radial and the axial directions, and the cosine distribution of the neutron flux along the axis of the annular reactor, expressions for the neutron flux were obtained in terms of a Fourier-Bessel series. The expression of the neutron current derived this way becomes negative infinity at both ends of the annular reactor. Numerical calculations based on Stummel's theory have been carried out by Aline (1) with the further simplified assumptions. The result seems to agree very well with the original calculation made by Stummel. In Aline and Stummel's work, the flux gradient at the surface of the void is assumed to be small compared to the value of the flux at the same point. The question of the validity of this assumption and thus the validity of the diffusion theory arises near both ends of the reactor where the flux gradient approaches negative infinity. It follows that there must be a physical range in which the diffusion theory will hold. The discussion of the validity of the streaming condition is one point of interest in this thesis and will be discussed further in the latter sections.

All of the works cited so far are concerned with the change in nuclear parameters and the flux variations along the surface of a channel in the cylindrical reactor. The axial flux distribution is assumed to satisfy the critical equation $\Delta\phi + B^2\phi = 0$; i.e., the cosine distribution is assumed in all these investigations. This restriction is, by no means, general in the study of the effect of cavities on neutron diffusion.

The investigation of the neutron flux variation along an empty cylindrical duct in a shield was first carried out by Roe (13). This problem is different from the studies cited previously in the sense that the flux exhibits as exponential behavior instead of a cosine distribution in the axial direction. In addition, the basic equation required as the condition of the streaming current along the duct involves an additional term due to the direct contribution of the unscattered neutrons coming from the source. The problem becomes much more complicated owing to this additional term. In his work, Roe introduced the Laplace transform technique as an alternative means of solving the streaming flux. Since the Laplace transform of each term involves functions of very complicated forms such as Hankel's function and Lommel-Weber function, the determination of the inverse transform presents a problem. However, Roe managed to give the solution of a restricted case in which the distance from the source is very large by taking the first few terms of the series. This problem is also one point of interest in this thesis. By some simplified approximations, it is possible to approximate the case when the distance from the source z is not too large.

An alternative method for determining the neutron flux attenuation in a cylindrical duct was suggested by Simon and Clifford (14). Expressions were given in terms of the albedo.

All the investigations cited so far are concerned only with the flux variation along the surface of the cavity for some special cases. Nothing has been said about the behavior of neutrons in the cavity and the localized flux variation in the medium due to the introduction of a cavity. It is the purpose of this thesis to present the theoretical investigation of the effect of cavities on neutron diffusion in a somewhat broader sense.

III. THEORY OF NEUTRON PROPAGATION IN THE ABSENCE OF SCATTERING, ABSORBING AND MULTIPLYING PROCESSES

The phenomenon of neutron propagation is somewhat analogous to the propagation of gas molecules. It is generally understood that neutrons tend to move from a region of high neutron density to one of low density and consequently give rise to a net neutron current. When there is a medium present, this phenomena is referred to as neutron diffusion. This diffusion phenomenon may, however, be complicated by reactions, such as scattering, absorbing and multiplying processes, with the medium in which neutrons diffuse. The behavior of neutrons in a medium can be, in general, described by the fundamental equation known as the Boltzmann equation. For mathematical simplicity, the energy dependence of the neutron density is usually neglected and all neutrons are assumed to have the same velocity. The one-velocity Boltzmann equation can be written in the form

$$\bar{\Omega} \cdot \text{grad } f(\bar{r},\bar{\Omega}) + \Sigma f(\bar{r},\bar{\Omega}) = \int \Sigma_s(\bar{\Omega}' \rightarrow \bar{\Omega}) f(\bar{x},\bar{\Omega}' \rightarrow \bar{\Omega}) d\bar{\Omega}' + S(\bar{r},\bar{\Omega}) \tag{1}$$

for the steady state, where $\bar{\Omega}$, Σ and Σ_s are the unit solid angle and the macroscopic total and scattering cross-section respectively.

The neutron distribution is characterized by the angular flux density $f(\bar{x},\bar{\Omega})$ in Weinberg and Wigner's (19) notation, which is the number of neutrons in the volume element $d\bar{r}$ around $\bar{r}$ and whose directions of motion lie in the solid angle $d\bar{\Omega}$ around $\bar{\Omega}$, multiplied by the speed of these neutrons. The first term in Eq. 1 describes the straight forward motion along $\bar{\Omega}$ and the direction of motion changes if and only if the neutron suffers a collision. The second term represents the total number of neutrons removed by the scattering and absorption of the medium per differential volume $d\bar{r}$ and solid angle $d\bar{\Omega}$. The first term on the right hand side of Eq. 1 corresponds to collisions in which the direction $\bar{\Omega}'$ is changed into $\bar{\Omega}$. The last term represents the source strength.

In particular, the steady state Boltzmann equation is reduced to the form

$$\bar{\Omega} \cdot \text{grad } f(\bar{r},\bar{\Omega}) = S(\bar{r},\bar{\Omega}) \tag{2}$$

in a vacuum, where the scattering and absorbing processes do not occur. Furthermore, if there is no source of neutrons present, it becomes

$$\bar{\Omega} \cdot \text{grad } f(\bar{r},\bar{\Omega}) = 0 \tag{3}$$

analogous to the continuity equation in fluid dynamics. Physically, Eq. 3 indicates that $f(\bar{r},\bar{\Omega})$ is constant along $\bar{\Omega}$. Hence, if an arbitrary coordinate s is selected in the direction of $\bar{\Omega}$, then

$$\frac{\partial f}{\partial s} = 0 \tag{4}$$

Consider a vacant region R bounded by any closed surface as shown in Fig. 1. The value of f is the same at all points along the unit vector $\bar{\Omega}$. Therefore, it is remarkable to note that once the incoming angular flux at A is known or the angular flux at any point along $\bar{\Omega}$ is specified, one is able to obtain f at any other point along $\bar{\Omega}$, i.e.,

$$f(A,\bar{\Omega}) = f(B,\bar{\Omega}) = f(C,\bar{\Omega}) \tag{5}$$

In particular, if the distribution on the boundary is isotropic, it will also be isotropic within the region R.

From a practical point of view, it is always desirable to express this angular flux in terms of the integrated flux $\phi(\bar{r})$. The integrated flux $\phi(\bar{r})$ is defined by

$$\phi(\bar{r}) = \int_{\bar{\Omega}} f(\bar{r},\bar{\Omega})\,d\Omega \tag{6}$$

To illustrate how angular flux is converted to integrated flux, assume that the outer region surrounding the vacant region R (Fig. 1) is a weakly absorbing medium in which the diffusion theory is valid. The angular flux at any point P in the outer medium far away from the boundary is given in the form (10)

$$f(P,\bar{\Omega}) = \frac{1}{4\pi}\left[\phi(P) + \frac{1}{\Sigma_s}\frac{\partial\phi(P)}{\partial\mu}\right] \tag{7}$$

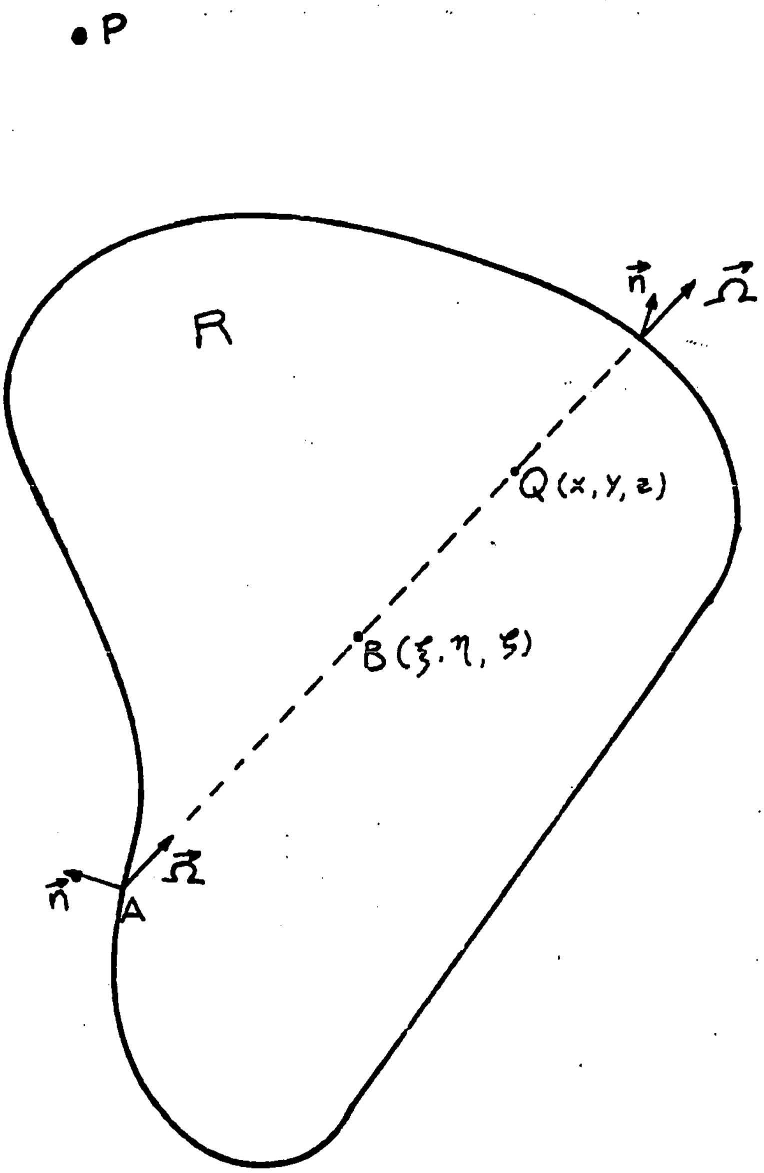


Fig. 1. Neutron streaming in a vacant region R

where $\frac{\partial}{\partial\mu}$ is the directional derivative opposite to $\bar{\Omega}$. This equation was derived on the assumption similar to the derivation of current density given by Glasstone and Edlund (8).

If the diffusion theory also holds at the boundary point, then

$$f(A,\bar{\Omega}) = \frac{1}{4\pi}\left[\phi(A) + \frac{1}{\Sigma_s} \frac{\partial\phi(A)}{\partial\mu} \right] \tag{8}$$

where the directional derivative $\frac{\partial}{\partial\mu}$, in general, can be divided into two tangential and one normal component:

$$\begin{aligned} \frac{\partial}{\partial\mu} &= -\bar{\Omega} \cdot t_1 \frac{\partial}{\partial t_1} - \bar{\Omega} \cdot t_2 \frac{\partial}{\partial t_2} - \bar{\Omega} \cdot \bar{n} \frac{\partial}{\partial n} \\ &= \bar{\Omega} \cdot \text{grad} \end{aligned} \tag{9}$$

The necessary condition for the diffusion theory to be valid is that the gradient of flux must be small. If the region R is small compared to the surrounding medium, it is reasonable to assume that the term $\frac{\partial\phi}{\partial\mu}$ in Eq. 8 is insignificant compared to the first term. Substituting Eq. 8 into Eq. 3 and neglecting the derivatives of the second order, one has

$$\bar{\Omega} \cdot \text{grad}\ \phi(\bar{x}) = 0 \tag{10}$$

The assumption that the diffusion theory holds on the boundary point is actually equivalent to the assumption that the flux on the boundary point of the region R approaches the isotropic distribution. By the definition of the integrated flux, the term "isotropic" means

$$f(\bar{x},\bar{\Omega}) = \frac{\phi(\bar{x})}{4\pi} \tag{11}$$

where ϕ is not dependent on $\bar{\Omega}$. It is approximately true according to Eq. 8 if $\frac{\partial\phi}{\partial\mu}$ is small compared to ϕ.

In most practical cases when one is only interested in qualitative results, this assumption is frequently made. In the present work, this assumption will be used for the case of a spherical cavity. However, it is not correct for a long cylindrical cavity. The discussion will be given in the following section.

The assumption that the flux distribution is isotropic on the boundary simplifies the problem considerably. Consider points $B(\xi,\eta,\zeta)$ and $Q(x,y,z)$ along $\bar{\Omega}$ in the region R as shown in Fig. 1. Let B be fixed and Q be arbitrary. Integrating Eq. 10 with respect to the surface of an arbitrary sphere with the radius equal to QB and applying the Divergence Theorem, one has

$$\begin{aligned}\int_{\text{Surface}} \bar{\Omega} \cdot \operatorname{grad} \phi(\bar{x})\,dS &= \int_{\text{Volume}} \operatorname{div}(\operatorname{grad} \phi(\bar{x}))\,dV \\ &= 0\end{aligned} \tag{12}$$

Since the integration is over an arbitrary volume, Eq. 12 is true if and only if the condition

$$\text{div (grad } \phi(\bar{x})) = \nabla^2 \phi(\bar{x}) = 0 \tag{13}$$

is satisfied, where ∇^2 is the Laplacian operator. Hence, the problem is reduced to the classical Dirichlet type of problem. This enables one to determine the flux in the region R if the value of the flux at the boundary is known.

IV. NEUTRON STREAMING IN A CYLINDRICAL CAVITY

Consider a cylindrical cavity contained in a large diffusing medium as shown in Fig. 2. Without losing generality, assume that there exists a uniform plane source located at $z = 0$ in a cylindrical coordinate with the axis of the cavity taken as the z axis. The length of the cylindrical cavity is assumed to be very long compared to the radius R and the diffusion length of the medium L.

Part of neutrons emitted from the source may stream through the air space unscattered until they reach the surface of the channel wall, whereas others may diffuse in the outer medium itself and may or may not enter the channel. Assume that there are no scattering and no absorption in air. Hence, the introduction of a cavity in a diffusing medium is equivalent to the introduction of a vacant region R as discussed in the preceding section. Since the length of the channel is large, the leakage of neutrons is not significant. The neutron flux variation along the wall is of interest. Once the flux variation along the wall is known, there will be no problem in determining the flux distribution within the cavity. This can be readily seen from the preceding discussion. However, conditions at the wall of the cavity are not easy to determine. The difficulties arise from the fact that the flux at any point on the wall depends on the flux at every other

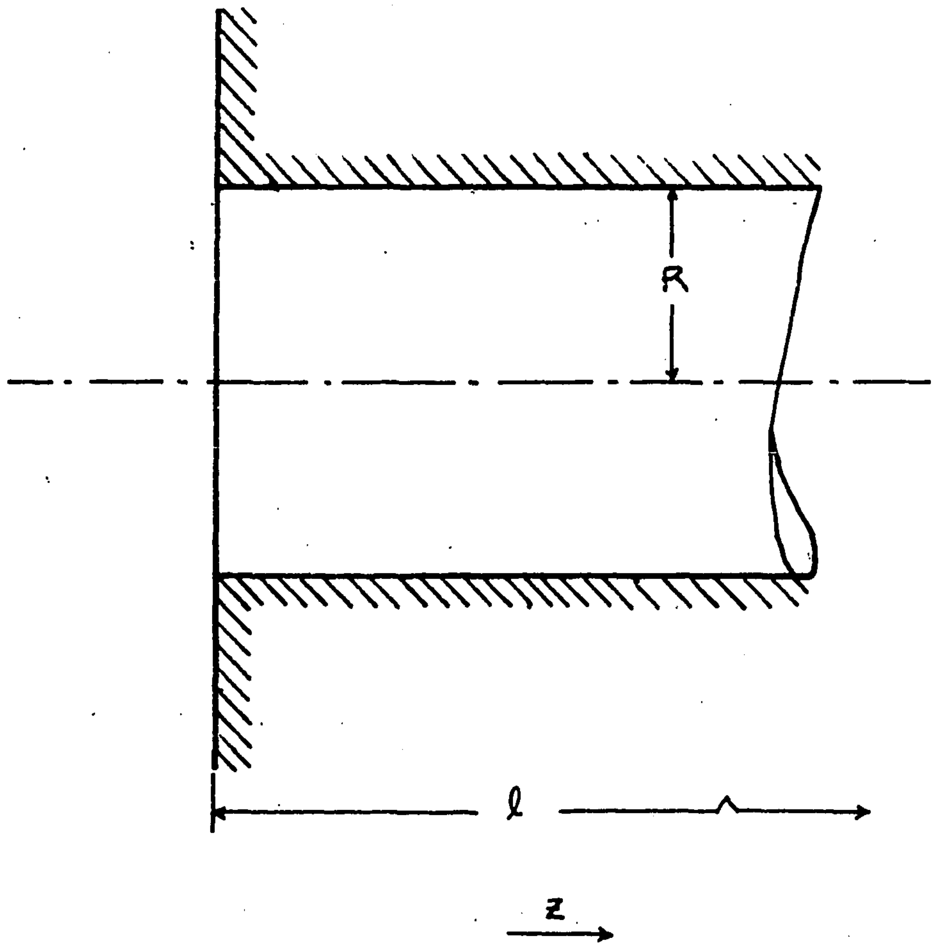


Fig. 2. A cylindrical channel in an infinite medium

point on the wall. It is, therefore, desirable to introduce the following assumptions so that an analytical solution to the flux variation is possible:

1. No neutron is created or absorbed within the cavity.
2. Scattering and absorbing processes take place only in the outer medium.
3. Diffusion theory is valid throughout the outer medium.
4. The neutrons are monoenergetic.

Consider first the case in which the uniform plane source of strength S emits thermal neutrons only. Note that the following equations derived to give the streaming conditions are equally applicable to the case in which the source is replaced by a plane fast neutron source, provided that proper parameters are used.

With these assumptions, the flux variation at the inner surface can be derived with the aid of the geometric relations. Consider any differential surface area dA on the cavity wall at (r,z) as shown in Fig. 3.

There are two ways by which neutrons can reach dA by streaming: (1) direct streaming from any point, say dA_1 on the source plane without being scattered; (2) indirect streaming from any other point on the wall, say dA_2 across the channel. By integrating the contribution of (1) and (2) over the surface area of the plane source within $r \leq R$ and the surface area of the channel respectively, one can obtain the total

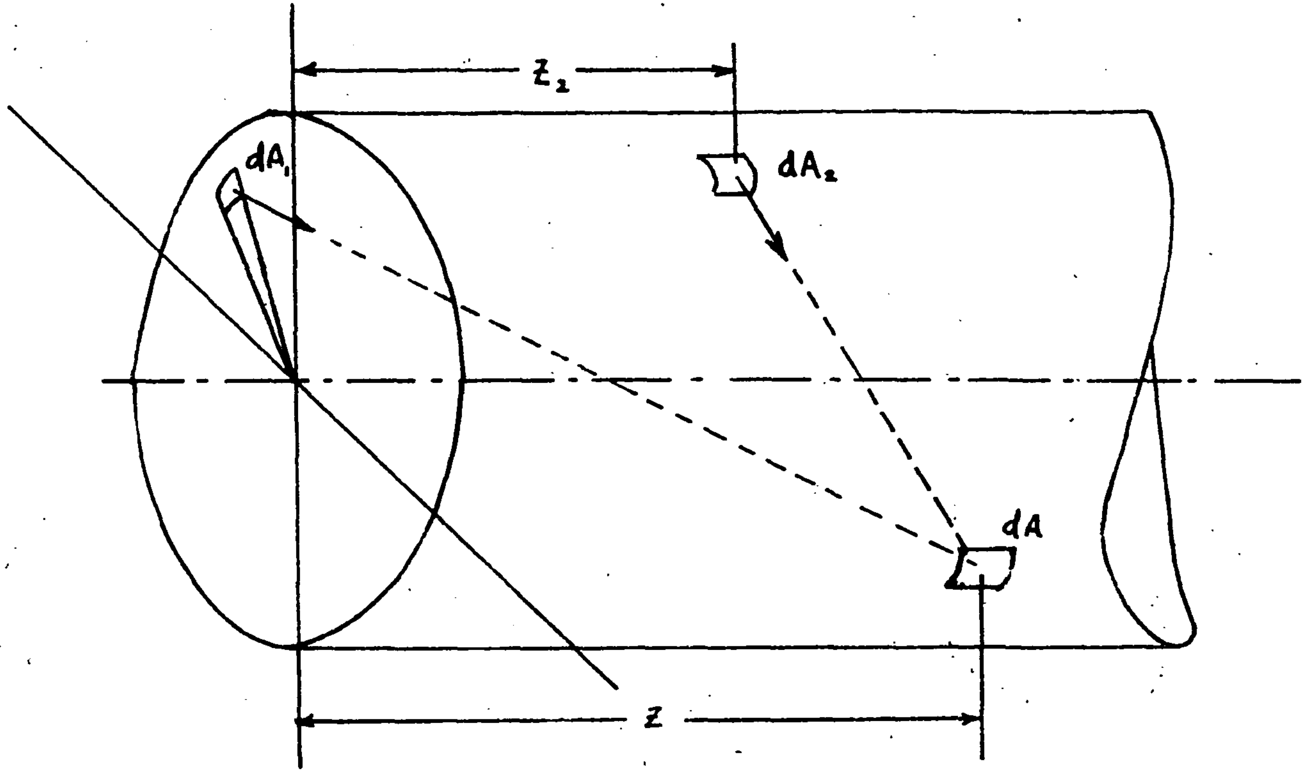


Fig. 3. Neutron streaming in a cylindrical channel

streaming current that eventually reaches dA at (r,z) through and across the channel.

A. Direct Streaming

The streaming current coming directly from the source through the air channel to reach dA will be considered first. The following relations can be obtained by referring to Fig. 4:

$$dA_1 = \rho \; d\rho \; d\alpha \tag{13}$$

where $\rho \leq R$ is any distance from the origin (0,0) on the source plane at z = 0 and α is the angle between f and the line OB.

The probability that neutrons enter the channel at dA_1 in the direction of $\bar{\Omega}$ per unit solid angle is

$$\frac{(\bar{n}_1,\bar{\Omega})}{\pi}$$

where $\bar{n}_1$ and $\bar{\Omega}$ are unit vectors specified in Fig. 4. The differential solid angle $d\Omega$ subtended by dA is

$$\frac{(-\bar{\Omega},\bar{n})\; dA}{E^2}$$

Hence, the probability of neutrons entering at dA_1 and reaching dA is

$$\frac{(\bar{n}_1,\bar{\Omega})}{\pi} \cdot \frac{(-\bar{\Omega},\bar{n})}{E^2}\; dA$$

and the total number of neutrons leaving the source plane that will reach dA without being scattered is

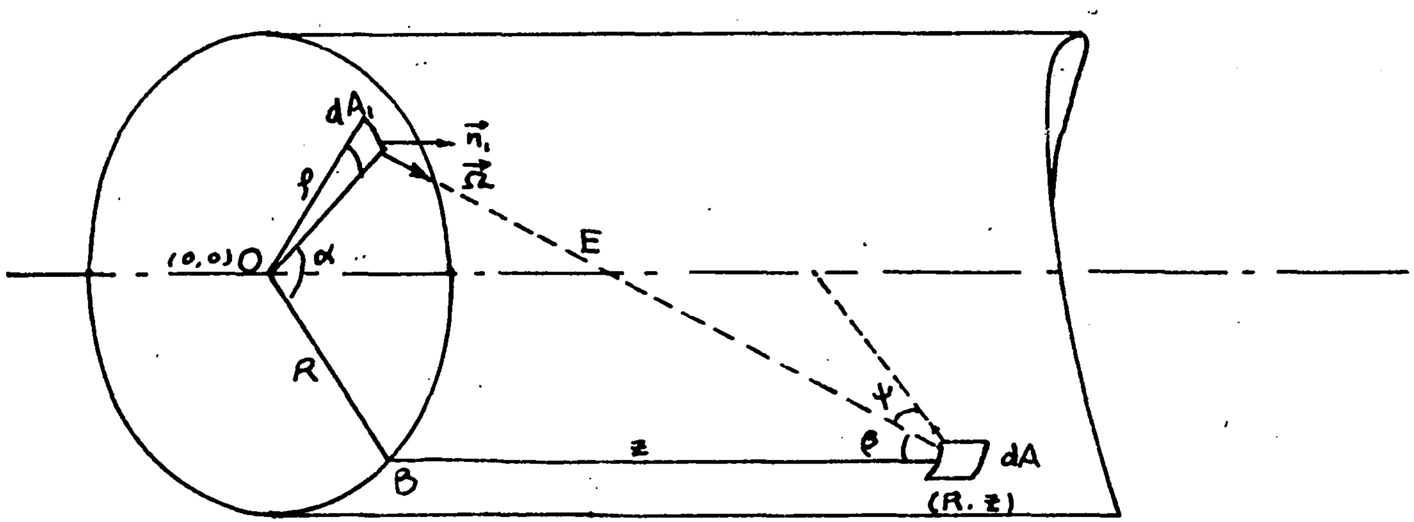


Fig. 4. Direct streaming in the cylindrical channel

$$N_1 = dA \int_{A_1} S \, \frac{(\bar{n}_1,\bar{\Omega})(-\bar{\Omega},\bar{n})}{\pi E^2} \, dA_1$$

$$= dA \int_0^{2\pi} \int_0^R \frac{S}{\pi E^2} \cos\beta \cos\psi \; \rho \, d\rho \, d\alpha \qquad (14)$$

where S designates the uniform source strength and quantities β, ψ and α are specified in Fig. 4.

From the geometrical relations shown in Fig. 4, it may be readily shown that

$$E^2 = z^2 + \rho^2 + R^2 - 2\rho R \cos\alpha \qquad (15)$$

$$\cos\beta = \frac{z}{E} \qquad (16)$$

$$\cos\psi = \frac{R - \rho\cos\alpha}{E} \qquad (17)$$

Thus, Eq. 14 becomes

$$N_1 = dA \; S \int_0^R \rho \, d\rho \int_0^{2\pi} \frac{z(R - \rho\cos\alpha)d\alpha}{\pi(z^2 + \rho^2 + R^2 - 2\rho R\cos\alpha)^2} \qquad (18)$$

This integral can not be evaluated directly. However, the evaluation of this integral may be easily carried out indirectly as follows. Consider the identity

$$\int_0^{2\pi} \frac{1}{\pi} \ln(z^2 + \rho^2 + R^2 - 2\rho R\cos\alpha)d\alpha \qquad (19)$$

$$= 2\left[\ln \frac{(z^2+\rho^2+R^2) + \sqrt{(z^2+\rho^2+R^2)^2 - 4\rho^2R^2}}{2}\right]$$

By applying Leibnitz's rule for the differentiation under the integral sign, one has

$$N_1 = -dA\ S\ \frac{1}{2}\frac{\partial^2}{\partial(z)\partial(2R)}\int_0^R \rho\ d\rho \int_0^{2\pi}\frac{1}{\pi}\ln(z^2+\rho^2+R^2-2\rho R\cos\alpha)\,d\alpha$$

$$= dA\ \frac{S}{2}\int_0^R \rho\ d\rho\ \frac{\partial^2}{\partial(z)\partial(2R)}\left[2\ \ln\frac{(z^2+\rho^2+R^2)+\sqrt{(z^2+\rho^2+R^2)^2-4\rho^2R^2}}{2}\right] \tag{20}$$

The differential term can be simplified to the form

$$\frac{\partial}{\partial z}\ \frac{\partial}{\partial(2R)}\left[2\ \ln\frac{(z^2+\rho^2+R^2+\ \sqrt{(z^2+\rho^2+R^2)^2-4\rho^2R^2}}{2}\right]$$

$$= \frac{\partial}{\partial z}\quad 2\ \frac{R\left[\sqrt{(z^2+\rho^2+R^2)^2-4\rho^2R^2}+(z^2+R^2+\rho^2-2\rho^2)\right]}{\sqrt{(z^2+\rho^2+R^2)^2-4\rho^2R^2}\left[(z^2+\rho^2+R^2)+\sqrt{(z^2+\rho^2+R^2)-4\rho^2R^2}\right]}$$

$$= \frac{\partial}{\partial z}\ \frac{1}{R}\left\{\frac{2R^2-(z^2+\rho^2+R^2)}{\sqrt{(z^2+\rho^2+R^2)^2-4\rho^2R^2}}-1\right\}\ x\ \frac{1}{2} \tag{21}$$

Let $u = z^2 + \rho^2 + R^2,\quad du = 2\ \rho\ d\rho$ (22)

The substitution of Eq. 21 and 22 into Eq. 20 gives

$$N_1 = -dA\ \frac{S}{2}\ \frac{\partial}{\partial z}\int_{z^2+R^2}^{z^2+2R^2}\frac{1}{2R}\left[\frac{-(u-2R^2)}{\sqrt{(u-2R^2)^2+4R^2z^2}}-1\right]du$$

$$= -dA\ S\ \frac{1}{4R}\ \frac{\partial}{\partial z}\left[z^2-\ \sqrt{z^4+4R^2z^2}\right]$$

$$= -dA\ \frac{S}{2R}\left[\frac{z^2+2R^2}{\sqrt{z^2+4R^2}}-z\right] \tag{23}$$

Let

$$P_1(z) = \frac{1}{R}\left[\frac{z^2 + 2R^2}{\sqrt{z^2 + 4R^2}} - z\right] \qquad (24)$$

The quantity $P_1(z)$ will be referred to as the direct streaming probability.

B. Indirect Streaming

The contribution of neutrons coming from any point (R, z_2) on the wall of the channel can be derived from a diffusion approximation. Consider any differential volume dV in the outer medium as shown in Fig. 5.

The number of collisions of neutrons per unit time that take place in dV is $\Sigma_s \phi(r', z')dV$. If the scattering is isotropic, the probability that a neutron in dV will be scattered in the direction such as to pass the differential area dA is given by

$$\frac{(-\bar{\Omega}, \bar{n})\,dA}{4\pi\rho^2}$$

where ρ is the distance between dV and dA. If the absorption cross section of the medium is negligible compared to the scattering cross section, the probability that neutrons having the direction of motion within $d\Omega$ will reach dA without further scattering is $\exp\left[-\Sigma_s(\rho - \delta)\right]$. Hence, the number of neutrons scattered from the volume element dV, which actually reach dA per unit time is given by

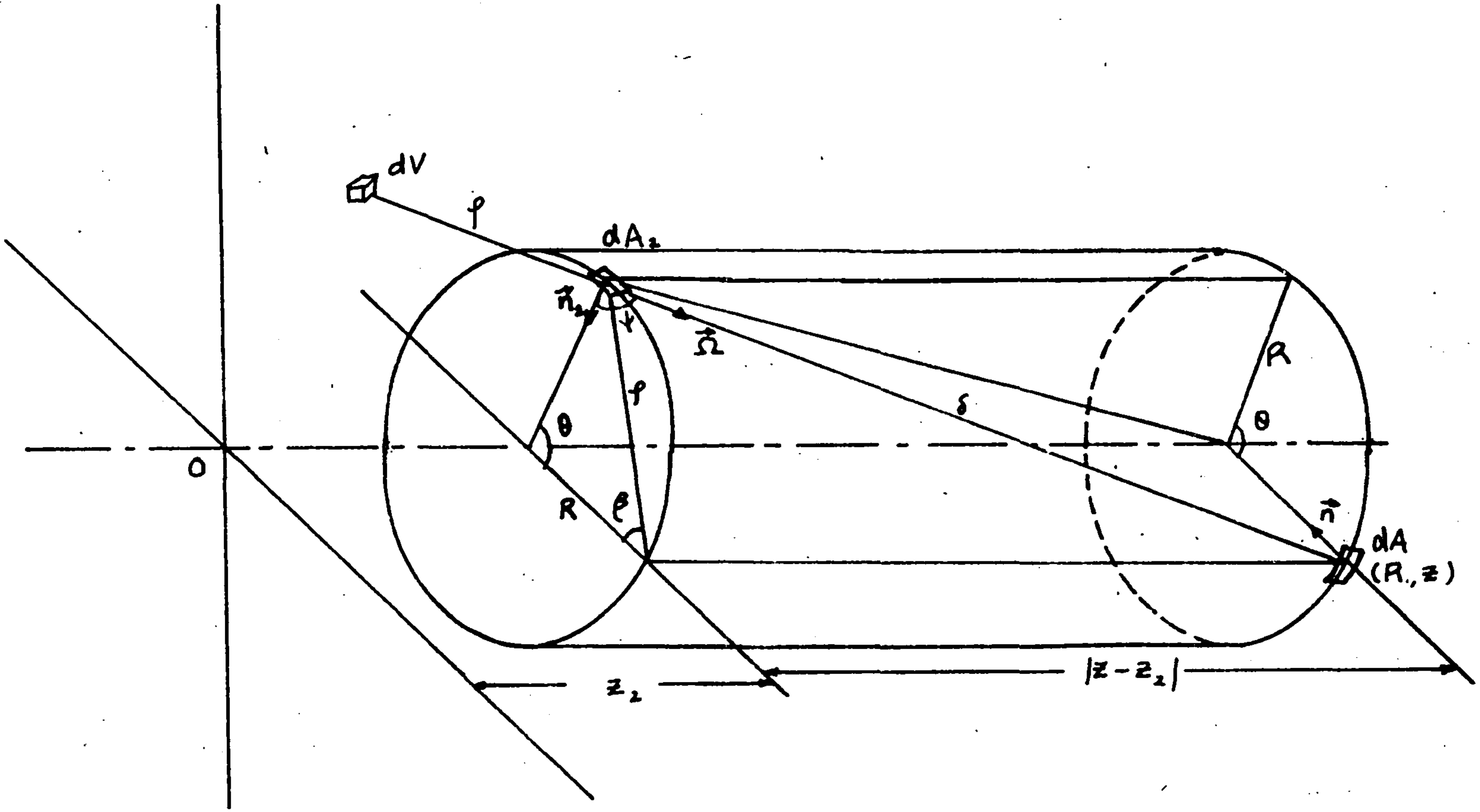


Fig. 5. Indirect streaming in the cylindrical channel

$$\Sigma_s \phi(r', z') dV \frac{(-\bar{\Omega}, \bar{n}) dA}{4\pi\rho^2} \exp\left[-\Sigma_s(\rho - \delta)\right]$$

Therefore, the total number of indirect streaming neutrons is

$$N_2 = dA \int_V \frac{\Sigma_s \phi(r', z')(-\bar{\Omega}, \bar{n}) \exp\left[-\Sigma_s(\rho - \delta)\right]}{4\pi\rho^2} dV \qquad (25)$$

The flux $\phi(r', z')$ at dV can be expanded into Taylor's series with respect to (R, z_2), whereby

$$\phi(r', z') = \{\phi - (\rho - \delta)\bar{\Omega} \cdot \text{grad } \phi + \cdots\}_{\text{at } (R, z_2)} \qquad (26)$$

If the medium under consideration is reasonably large, Eq. 25 can be approximated by

$$N_2 = dA \frac{\Sigma_s}{4\pi} \int_V \{\phi(R, z_2) - (\rho-\delta)\bar{\Omega} \cdot \text{grad } \phi(R, z_2)\} \cdot \frac{(-\bar{\Omega}, \bar{n})}{\rho^2} \exp\left[-\Sigma_s(\rho-\delta)\right] dV \qquad (26')$$

From geometrical relationships, the differential volume dV may be replaced by

$$dV = \rho^2 \, d\rho \, d\Omega.$$

Hence, Eq. 26' becomes

$$N_2 = dA \frac{\Sigma_s}{4\pi} \int_\delta \exp\left[-\Sigma_s(\rho-\delta)\right] d\rho \int_\Omega \{\phi(R, z_2) - (\rho-\delta)\bar{\Omega} \cdot \text{grad } \phi(R, z_2)\}(-\bar{\Omega}, \bar{n}) d\Omega \qquad (27)$$

in which it is assumed that the extension of the diffusing medium is very large compared to the mean free path λ_s so that the integration over ρ can be extended to infinity. Furthermore, $d\Omega$ is related to dA_2 by the following relationship:

$$d\Omega = \frac{dA_2(\bar{\Omega},\bar{n}_2)}{\delta^2} = \frac{R\,dz_2\,d\theta(\bar{\Omega},\bar{n}_2)}{\delta^2} \qquad (28)$$

Substituting Eq. 28 into Eq. 27 and integrating, one has

$$N_2 = dA\,\frac{R}{4\pi}\int_0^{\ell} dz_2 \int_0^{2\pi} \{\phi(R,z_2) - \frac{1}{\Sigma_s}\,\bar{\Omega}\cdot\text{grad}\,\phi(R,z_2)\}\cdot \frac{(-\bar{\Omega},\bar{n})(\bar{\Omega},n_2)}{\delta^2}\,d\theta \qquad (29)$$

By referring to Fig. 5, one can write Eq. 29 in a integrable form with the aid of the following geometrical relations. It is readily seen that

$$\mathscr{S} = 2R\sin\beta; \quad \beta = \theta/2 \qquad (30)$$

$$\delta^2 = (|z - z_2|)^2 + (2R\sin\frac{\theta}{2})^2 \qquad (31)$$

$$\xi^2 = R^2 + |z - z_2|^2 = R^2 + \delta^2 - 2R\delta\cos\psi \qquad (32)$$

$$\cos\psi = \frac{2R}{\delta}\sin^2\frac{\theta}{2} = \frac{R}{\delta}(1 - \cos\theta) \qquad (33)$$

$$(\bar{\Omega},\bar{n}_2) = (-\bar{\Omega},\bar{n}) = \cos\psi \qquad (34)$$

The substitution of Eq. 31, 32 and 34 into Eq. 29 gives

$$N_2 = dA\,\frac{R^3}{4\pi}\int_0^{\ell} dz_2\phi(R,z_2)\int_0^{2\pi}\frac{(1-\cos\theta)^2 d\theta}{[\,|z-z_2|^2 + 2R^2(1-\cos\theta)\,]^2} - \frac{R^3}{4\pi}$$

$$\int_0^{\ell} dz_2 \int_0^{2\pi}\bar{n}\cdot\text{grad}\,\phi(R,z_2)\frac{(1-\cos\theta)^2 d\theta}{[\,|z-z_2|^2+2R^2(1-\cos\theta)\,]^2} \qquad (35)$$

The absolute sign is imposed because $|z - z_2|$ is considered to be the magnitude of the relative distance between dA and dA_2 along the z-axis. The second integral vanishes if Eq. 10 is applicable. Without losing generality, this integral remains in Eq. 35. The condition for which Eq. 10 is applicable will be discussed later.

To evaluate the first integral with respect to θ, one needs to consider first an integral of the form

$$I_1 = \frac{1}{\pi}\int_0^{2\pi} \ln\,[\,|z-z_2|^2 + 2R^2(1-\cos\theta)\,]\,d\theta$$

$$= 2\{\ln\frac{|z-z_2|^2 + 2R^2 + |z-z_2|\sqrt{|z-z_2|^2 + 4R^2}}{2}\} \qquad (36)$$

Again, according to Leibnitz's rule, one has

$$\int_0^{2\pi}\frac{(1-\cos\theta)^2 d\theta}{\pi[\,|z-z_2|^2 + 2R^2(1-\cos\theta)\,]^2} = -\frac{\partial^2 I_1}{\partial(2R^2)^2} \qquad (37)$$

By direct differentiation, one has

$$\frac{\partial I_1}{\partial(2R^2)} = \frac{2\sqrt{|z-z_2|^2 + 4R^2} + 2|z-z_2|}{\big[(|z-z_2|^2 + 2R^2) + |z-z_2|\sqrt{|z-z_2|^2 + 4R^2}}$$

$$= \frac{1}{2}\left[\frac{\sqrt{|z-z_2|^2 + 4R^2} - |z-z_2|}{R^2\sqrt{|z-z_2|^2 + 4R^2}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{R^2} - \frac{|z-z_2|}{R^2\sqrt{|z-z_2|^2 + 4R^2}}\right] \tag{38}$$

Hence,

$$-\frac{\partial^2 I_1}{\partial(2R^2)^2} = \frac{1}{2R^4}\left\{1 - \frac{|z-z_2|^3 + 6R^2|z-z_2|}{\left[|z-z_2|^2 + 4R^2\right]^{3/2}}\right\}$$

$$= \frac{1}{2R^4}\left\{1 - \frac{\dfrac{|z-z_2|^3}{R^3} + \dfrac{6|z-z_2|}{R}}{\left[\dfrac{|z-z_2|^2}{R^2} + 4\right]^{3/2}}\right\} \tag{39}$$

Let

$$P_2(|z-z_2|) = \frac{1}{2R}\left\{1 - \frac{\dfrac{|z-z_2|^3}{R^3} + \dfrac{6|z-z_2|}{R}}{\left[\dfrac{|z-z_2|^2}{R^2} + 4\right]^{3/2}}\right\} \tag{40}$$

which will be referred to as indirect streaming probability.

Eq. 35 thus becomes

$$N_2 = dA \int_0^{\rho} \frac{\phi(R, z_2)}{4} P_2(|z-z_2|) dz_2$$

$$- \frac{R^3}{4\pi} \int_0^{\rho} dz_2 \int_0^{2\pi} \{\bar{\Omega} \cdot \text{grad } \phi(R, z_2) \frac{(1 - \cos \theta)^2 d\theta}{[\,|z-z_2|^2 + 2R^2(1-\cos \theta)\,]^2}\} \tag{41}$$

The last integral will be discussed in the following section.

C. Fundamental Equation of Neutron Streaming in a Cylindrical Channel

The balance of neutrons at an arbitrary point on the wall of a cylindrical channel contained in a large diffusing medium can be readily seen from Fig. 3. The partial neutron current at dA is the well known result which can be obtained by a method similar to that used to derive Eq. 26.

$$J_- = \frac{\phi(R, z)}{4} + \frac{1}{6\Sigma_s} (\bar{n}, \text{ grad } \phi(R, z)) \tag{42}$$

where the - sign designates the inward flow across dA. This is, in fact, the well known partial current based on diffusion theory in the absence of the channel. The net current entering the channel across dA is

$$j = \frac{1}{3\Sigma_s} \frac{\partial \phi(R, z)}{\partial r} \tag{43}$$

A combination of Eq. 23, 41, 42 and 43 gives the net number of neutrons that enters the channel across dA as

$$\frac{1}{3\Sigma_s}\frac{\partial \phi(R,z)}{\partial r}\,dA = dA\;\frac{\phi(R,z)}{4} + \frac{1}{6\Sigma_s}(n,\ \text{grad}\ \phi(R,z)$$

$$- N_1 - N_2 \tag{44}$$

so that

$$\frac{\phi(R,z)}{4} - \frac{1}{6\Sigma_s}\frac{\partial \phi(R,z)}{\partial r} = \frac{S}{2R}\left[\frac{z^2 + 2R^2}{z^2 + 4R^2} - z\right]$$

$$+ \int_0^{\rho} \frac{\phi(R,z_2)}{8R}\left[1 - \frac{\frac{|z-z_2|^3}{R^3} + \frac{6|z-z_2|}{R}}{\left[\frac{|z-z_2|^2}{R^2} + 4\right]^{3/2}}\right]dz_2 \tag{45}$$

$$- \frac{R^3}{4\pi}\int_0^{\rho} dz_2 \int_0^{2\pi} \bar{\Omega}\cdot\text{grad}\ \phi(R,z_2)\frac{(1 - \cos\theta)^2 d\theta}{\left[|z-z_2|^2 + 2R^2(1-\cos\theta)\right]^2}$$

Equation 35 will be referred to as the Fundamental Equation for Neutron Streaming in a Cylindrical Channel. The analytical solution of an intego-differential equation of this form is rather impractical unless some simplifications are made.

It has been noted previously that if the dimension of the cavity is very small compared to that of the medium surrounding it, it is reasonable to assume an isotropic distribution on the boundary. It follows that the flux satisfies the continuity equation of the form defined by Eq. 10. From a practical point of view, this assumption is very helpful when one is only interested in a qualitative result. The validity of this assumption depends on the relative value of the flux gradient to that of the flux at the boundary point. This relative value, in turn, depends not only on the relative size of the cavity to that of the medium but also on the geometry of the cavity. Consider, for example, a small spherical cavity enclosed in a large non-multiplying and weakly absorbing medium. Here, Eq. 10 is a good approximation since there is no reason why a sharp change in neutron flux should occur. The case of the spherical cavity will be discussed further in the later section.

However, so far as the cylindrical channel is concerned, Eq. 10 is not satisfied in general. It may be a good approximation in the limit of small z, i.e., far away from the end of the medium. Since the leakage becomes significant near the end of the channel, the scattering is highly preferential in both the z and the r direction so that the flux gradient may eventually approach infinity at the end of the channel. This end effect has been discussed by Stummel (16) for annular reactors. In the present work, one is interested only in locations away from the end and the end effect will not be discussed in detail.

It is interesting to compare the relative magnitude of each term in Eq. 45. The results will enable one to see how the flux gradient varies as the function of z along the channel for large z. This leads to the conclusion that the diffusion approximation eventually breaks down at large z.

1. <u>Variation of the flux gradient at large z</u>

The first term on the right hand side of Eq. 45 is a rapidly decreasing function of z. The direct streaming probability $P_1(z) = \frac{1}{R}\left[\frac{z^2 + 2R^2}{\sqrt{z^2 + 4R^2}} - z \right]$ is plotted vs. z/R given in Fig. 6. The shape of this curve immediately suggests its similarity to curves of exponential nature. It is, therefore, possible to replace the hyper-geometric function of the form $\frac{1}{R}\left[\frac{z^2 + 2R^2}{\sqrt{z^2 + 4R^2}} - z \right]$ by the linear combination of two exponential curves analogous to the analysis of composite decay curves in radiochemistry. From Fig. 6, one can write

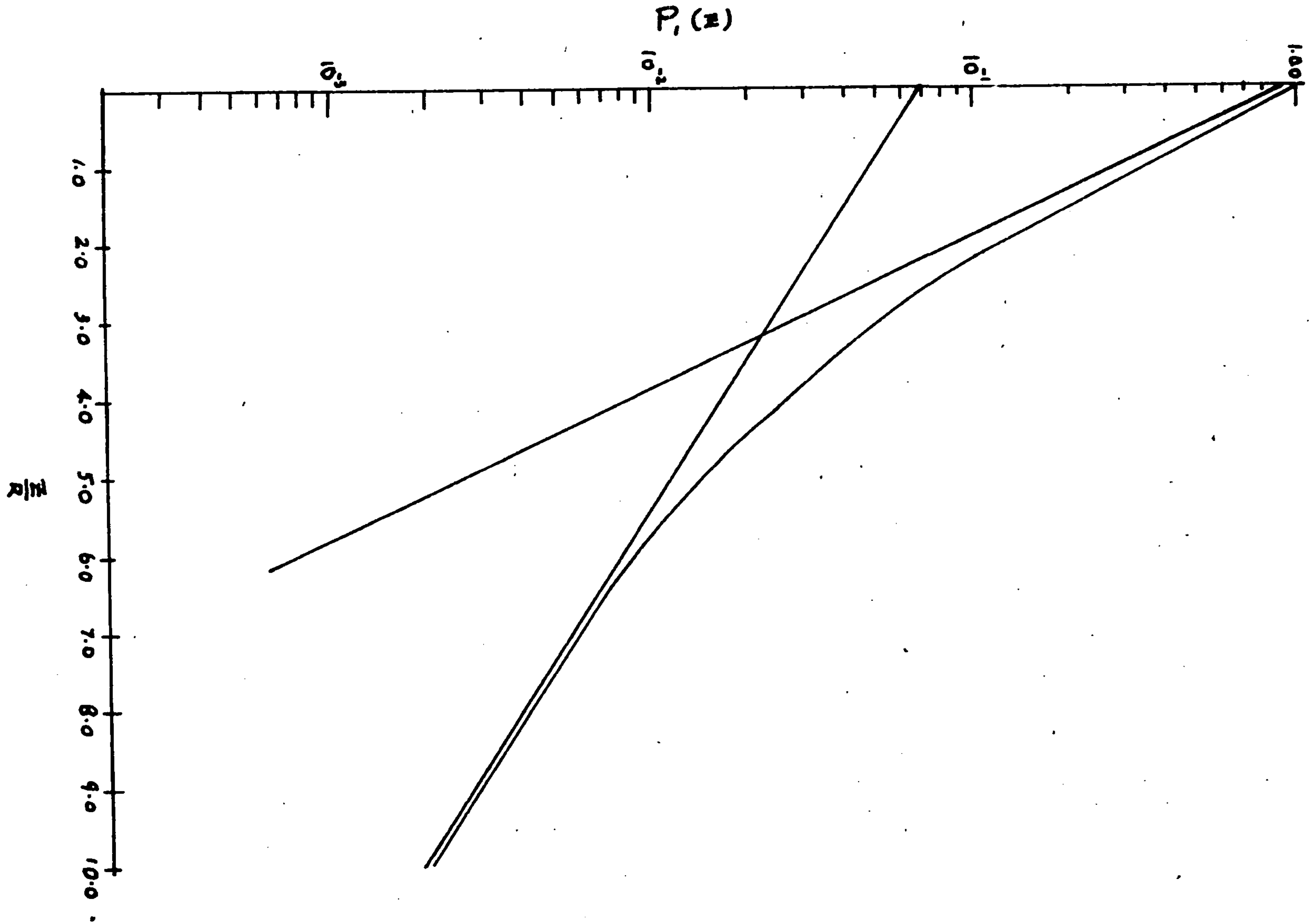
$$P_1(z) \cong 0.930 \exp\left[- \frac{1.187}{R} z \right] + 0.070 \exp\left[- \frac{0.365}{R} z \right] \tag{46}$$

in the range of $z/R \leq 10$. It is practically zero for $z/R \gg 10$.

Similarly, the indirect streaming probability $P_2(|z-z_2|)$ for diffusing neutrons can be represented by

$$P_2(|z-z_2|) = \left\{ 1 - \frac{\frac{|(z-z_2)^3|}{R^3} + \frac{6|z-z_2|}{R}}{\left[\frac{(z-z_2)^2}{R^2} + 4 \right]^{3/2}} \right\}$$

Fig. 6. Direct streaming probability vs. the distance from the source



$P_1(x)$
10^{-3}
10^{-2}
10^{-1}
1.00
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
$\frac{x}{R}$

$$\cong 0.955 \exp\left[-\frac{1.187}{R}|z-z_2|\right] + 0.045 \exp\left[-\frac{0.447}{R}|z-z_2|\right] \tag{47}$$

in the range of $\frac{|z-z_2|}{R} \leq 10$ as given in Fig. 7.

Again, the function decreases rapidly as $\frac{|z-z_2|}{R}$ increases. Hence, the kernel $P(|z-z_2|)$ in the integral of Eq. 45 acts like a Dirac δ-function and it has a very sharp peak at $z = z_2$. Consequently, it is possible to write

$$\int_0^{\ell} \frac{\phi(R, z_2)}{8R} P_2(|z-z_2|)dz_2 = \frac{\phi(R,z)}{8R}\int_0^{\ell}\left\{1 - \frac{\frac{|z-z_2|^3}{R^3} + \frac{6|z-z_2|}{R}}{\left[\frac{|z-z_2|^2}{R^2} + 4\right]^{3/2}}\right\}dz_2$$

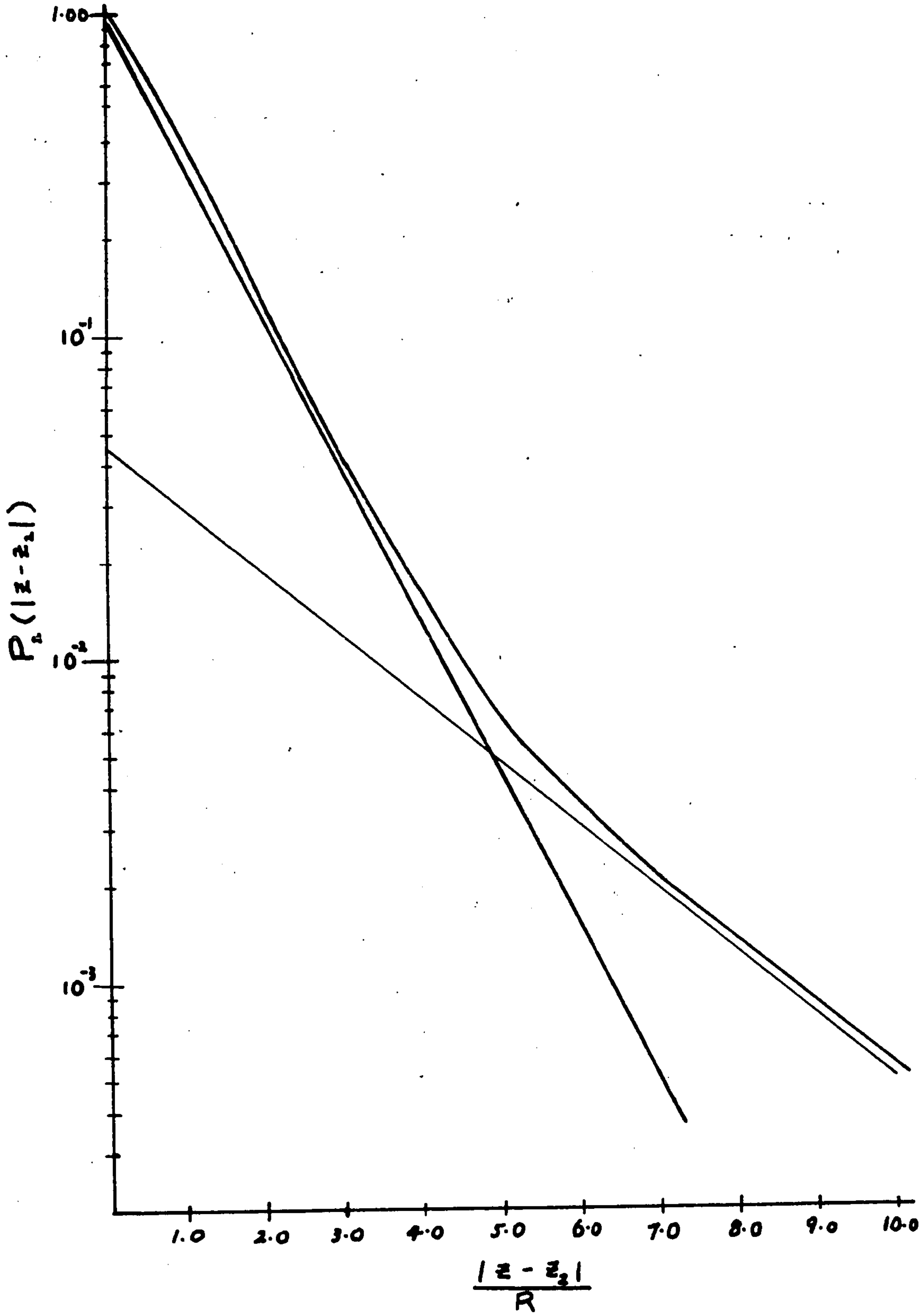
$$= \frac{\phi(R,z)}{8R} \left\{ - \left[\int_0^z \frac{\frac{(z-z_2)^3}{R^3} + \frac{6(z-z_2)}{R}}{\left[\frac{(z-z_2)^2}{R^2} + 4\right]^{3/2}} dz_2 \right.\right.$$

$$\left.\left. + \int_z^{\ell} \frac{\frac{(z_2-z)^3}{R^3} + \frac{6(z_2-z)}{R}}{\left[\frac{(z-z_2)^2}{R^2} + 4\right]^{3/2}} dz_2 \right] \right\}$$

$$= \frac{\phi(R,z)}{8R} \left\{ - \left[\frac{6R}{\sqrt{x^2+4}} - R\left(\sqrt{x^2+4} + \frac{4}{\sqrt{x^2+4}}\right)\right]_{z/R}^{0} \right.$$

$$\left. - \left[\frac{-6R}{\sqrt{y^2+4}} + R\left(\sqrt{y^2+4} + \frac{4}{\sqrt{y^2+4}}\right)\right]_0^{\frac{\ell-z}{R}} \right\}$$

Fig. 7. Indirect streaming probability vs. $\frac{|z-z_2|}{R}$


1.00
10^{-1}
10^{-2}
10^{-3}
$P_2(|z-z_2|)$
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
$\frac{|z-z_2|}{R}$

$$= \frac{\phi(R,z)}{8R} \left\{ \ell + 2R - \frac{z^2+2R^2}{\sqrt{z^2+4R^2}} - \frac{(\ell-z)^2+2R^2}{\sqrt{(\ell-z)^2+4R^2}} \right\} \tag{48}$$

This quantity reduces to $\frac{\phi(R,z)}{4}$ in the limit of $\frac{z}{R} \gg 1$ and $\frac{\ell-z}{R} \gg 1$, i.e.

$$\int_0^\ell P_2(|z-z_2|) \frac{\phi(R,z_2)}{8R} \simeq \frac{\phi(R,z)}{4} \tag{49}$$

To evaluate the last integral of Eq. 45, one needs to know $\bar{\Omega} \cdot \text{grad}\ \phi(R,z_2)$ first. In the cylindrical coordinate with axial symmetry

$$\bar{\Omega}\cdot\text{grad}\ \phi(R,z_2) = -\frac{(z-z_2)}{\delta}\frac{\partial\phi(R,z_2)}{\partial z} + \cos\psi\frac{\partial\phi(R,z_2)}{\partial r}$$

where all quantities are defined in Eq. 30 through 33 and in Fig. 5.

Hence, the last integral of Eq. 45 becomes

$$\frac{R^3}{4\pi}\int_0^\ell dz_2 \int_0^{2\pi} \frac{1}{\Sigma_s}\bar{\Omega}\cdot\text{grad}\ \phi(R,z_2)\frac{(1-\cos\theta)^2 d\theta}{[\,|z-z_2|^2+4R^2\,]^2}$$

$$= -\frac{R^3}{4\pi}\int_0^\ell dz_2 \int_0^{2\pi} \frac{1}{\Sigma_s}\frac{\partial\phi(R,z_2)}{\partial z_2}\frac{(z-z_2)(1-\cos\theta)^2}{[\,(z-z_2)^2+4R^2\,]^{5/2}} \tag{50}$$

$$+ \frac{R^4}{4\pi}\int_0^\ell dz_2 \int_0^{2\pi} \frac{1}{\Sigma_s}\frac{\partial\phi(R,z_2)}{\partial r}\frac{(1-\cos\theta)^3}{[\,(z-z_2)^2+4R^2\,]^{5/2}}\,d\theta$$

The integrals with respect to θ in Eq. 50 are related to the complete and associated complete elliptic functions and do not have solutions in analytical form. As is shown in the Appendix, they may be represented approximately by the following form:

$$\frac{R^3}{4\pi}\int_0^{\ell} dz_2 \int_0^{2\pi} \frac{1}{\Sigma_s}\,\bar{\Omega}\cdot \mathrm{grad}\,\phi(R,z_2)\frac{(1-\cos\theta)^2\,d\theta}{\left[(z-z_2)^2+4R^2\right]^2}$$

$$= -\frac{12.55R^3}{4\pi\Sigma_s}\int_0^{\ell} dz_2\left[\frac{\partial\phi(R,z_2)}{\partial z_2}\,\frac{(z-z_2)}{\left[(z-z_2)^2+4R^2\right]^{5/2}}\right]$$

$$+\frac{9.15R^4}{4\pi\Sigma_s}\int_0^{\ell} dz_2\left[\frac{\partial\phi(R,z_2)}{\partial z_2}\,\frac{1}{\left[(z-z_2)^2+4R^2\right]^{5/2}}\right] \tag{51}$$

The function $\frac{1}{\left[(z-z_2)^2+4R^2\right]^{5/2}}$ is again a rapidly decreasing function of $|z-z_2|$ as given in the Appendix and has a sharp peak at $z = z_2$. By using the preceding argument and straight forward integration, one has

$$\frac{9.15R^4}{4\pi\Sigma_s}\int_0^{\ell} dz_2\,\frac{\partial\phi(R,z_2)}{\partial r}\,\frac{1}{\left[(z-z_2)^2+4R^2\right]^{5/2}}$$

$$= \frac{9.15R^4}{4\pi\Sigma_s}\left[\frac{z}{12R^2(z^2+4R^2)^{3/2}}+\frac{z}{24R^4\sqrt{z^2+4R^2}}\right.$$

$$\left.-\frac{(z-\ell)}{24R^2\sqrt{(\ell-z)^2+4R^2}}-\frac{(z-\ell)}{12R^2\sqrt{(z-\ell)^2+4R^2}}\right]\frac{\partial\phi(R,z)}{\partial r}$$

$$\cong \frac{9.15}{4\pi\Sigma_s}\left(\frac{1}{12R^2z^2}\right)\frac{\partial\phi(R,z)}{\partial r} \tag{52}$$

in the limit of $z/R >> 1$; and $\frac{\ell-z}{R} >> 1$.

The mean value of the integral containing $\frac{\partial\phi(R,z_2)}{\partial z_2}$ can be determined by assuming $\nabla^2\phi(r,z) = \frac{1}{L^2}\,\phi(r,z)$ and $\frac{\partial\phi(R,0)}{\partial z} \cong -\frac{S}{2D}$.

Therefore,

$$\frac{12.55R^3}{4\pi\Sigma_s}\int_0^{\ell} dz_2\,\frac{\partial\phi(R,z_2)}{\partial z_2}\,\frac{(z-z_2)}{\left[(z-z_2)^2+4R^2\right]^{5/2}}$$

$$= \frac{2}{3}\left[\frac{\partial\phi(R,z_2)}{\partial z_2}\,\frac{1}{\left[(z-z_2)^2+4R^2\right]^{3/2}}\right]_0^{\ell}$$

$$-\frac{1}{L^2}\,\frac{2}{3}\,\phi(R,z)\left[\frac{(z-z_2)}{4R^2\sqrt{(z-z_2)^2+4R^2}}\right]_0^{\ell}\quad\frac{12.55}{4\pi\Sigma_s}$$

$$\cong \frac{2}{3}\left[\frac{-S}{2D}\,\frac{1}{z^3}\right]-\frac{1}{L^2}\,\frac{2}{3}\,\phi(R,z)\left[\frac{1}{2R^2}\right]\quad\frac{12.55}{4\pi\Sigma_s} \tag{53}$$

Hence, in the limit of $z >> R$, and $\ell - z >> R$, the $P_1(z)$ vanishes and Eq. 45 becomes

$$\frac{\phi(R,z)}{4}-\frac{1}{6\Sigma_s}\,\frac{\partial\phi(R,z)}{\partial r} \cong \frac{\phi(R,z)}{4}+\frac{12.55}{4\pi\Sigma_s}\quad\frac{2}{3}\left[\frac{-S}{2D}\cdot\frac{R^3}{z^3}\right]$$

$$-\frac{1}{L^2}\,\frac{2}{3}\,\phi(R,z)\left[\frac{1}{2R^2}\right]\quad-\frac{9.15R^4}{4\pi\Sigma_s}\left[\frac{1}{12R^2z^2}\right]\frac{\partial\phi(R,z)}{\partial r} \tag{54}$$

so that

$$\frac{\frac{\partial\phi(R,z)}{\partial r}}{\phi(R,z)} \cong \frac{\frac{R}{6L^2\Sigma_s} - \frac{R^3}{3L\Sigma_s}\,0(\frac{1}{z^3})}{-\frac{1}{6\Sigma_s}\,(1 - \frac{9.15R^2}{8\pi z^2})}$$

$$= \frac{\frac{R}{L^2} - \frac{2R^3}{L\Sigma_s}\,0(\frac{1}{z^3})}{-(1 - \frac{9.15R^2}{8\pi z^2})} \qquad (55)$$

where O(X) means the order of X.

This last equation implies that the applicability of the diffusion theory depends upon the value of the radius of the cavity R relative to the diffusion length of the medium L and the value z/L. In other words, if R is small compared to L and z/L is small, i.e., far away from the boundary, the ratio $\frac{\frac{\partial\phi(R,z)}{\partial r}}{\phi(R,z)}$ will be reasonably small. On the other hand, the diffusion theory breaks down in the vicinity of the channel if these conditions are not satisfied. Physically, the conclusion is rather obvious since the leakage of neutrons into the channel is more significant than neutrons, leaving the channel when z is large.

2. Simplified form of the fundamental equation

In most practical problems, one is interested only in the flux distribution at small values of z where the significant increase from its original value in the homogeneous system may be expected due to the presence of the cavity. The direct and indirect streaming terms in Eq. 45 predominate the terms containing the flux gradient. It has been mentioned previously that terms $P_1(z)$ and $P_2(|z-z_2|)$ can be approximated by the linear combination of two exponentials given by Eq. 46 and Eq. 47 in the reasonably large range of z. The integrands containing the flux gradient terms can also be approximated by exponential terms given in the Appendix. Hence, the Fundamental Equation can be written as

$$\frac{\phi(R,z)}{4} - \frac{1}{6\Sigma_s}\frac{\partial\phi(R,z)}{\partial r} = \frac{S}{2}\left[0.930\exp\left(-\frac{1.187}{R}z\right) + 0.07\exp\left(-\frac{0.365}{R}z\right)\right] + \int_0^{\ell}\frac{\phi(R,z_2)}{8R}\left[0.955\cdot\exp\left(-\frac{1.187}{R}|z-z_2|\right) + 0.045\exp\left(-\frac{0.447}{R}|z-z_2|\right)\right]dz_2$$

$$+ \frac{12.55}{4\pi\Sigma_s}\int_0^{\ell} dz_2\,\frac{\partial\phi(R,z_2)}{\partial z_2}\,\frac{(z-z_2)}{R}\left[0.062\exp\left(\frac{-1.31|z-z_2|}{R}\right) + 0.002\exp(-0.542|z-z_2|)\right]$$

$$- \frac{9.15R^4}{4\pi\Sigma_s} \int_0^{\ell} dz_2 \frac{\partial\phi(R,z_2)}{\partial r} \Big[0.062 \exp(-1.31 \; |z-z_2|) + 0.002 \exp(-0.542 \; |z-z_2|) \Big] \tag{56}$$

Note that the necessary condition for the diffusion theory to be valid is that $\frac{\partial\phi}{\partial r}/\phi$ and $\frac{\partial\phi}{\partial z}/\phi$ are small. If these conditions are satisfied for small values of R and in the reasonable range of z, the last two integrals are vanishingly small compared to the first integral in the range of validity, so that Eq. 56 becomes

$$\begin{aligned} \frac{\phi(R,z)}{4} - \frac{1}{6\Sigma_s} \frac{\partial\phi(R,z)}{\partial r} = \frac{S}{2}\Big[& 0.930 \exp(- \frac{1.187}{R} z) \\ & + 0.07 \exp(- \frac{0.365}{R} z) \Big] \\ & + \int_0^{\ell} \frac{\phi(R,z_2)}{8R} \Big[0.955 \exp(- \frac{1.187}{R} \; |z-z_2|) \\ & + 0.045 \exp(- \frac{0.447}{R} \; |z-z_2|) \Big] \end{aligned} \tag{57}$$

which is a type of the Wiener-Hapf type singular integral equation of the second kind if ℓ approaches infinity.

Eq. 57 can be further simplified by a simple physical argument. The original integral representing the contribution of the indirect streaming neutrons may be written in the form

$$\frac{1}{8R}\int_0^{\ell}\phi(z_2)\left\{1-\frac{\frac{|z-z_2|^3}{R^3}+\frac{6|z-z_2|}{R}}{\left[\frac{|z-z_2|^2}{R^2}+4\right]^{3/2}}\right\}dz_2$$

$$=\frac{1}{8R}\int_0^{z}\phi(R,z_2)\left\{1-\frac{\frac{(z-z_2)^3}{R^3}+\frac{6(z-z_2)}{R}}{\left[\frac{(z-z_2)^2}{R^2}+4\right]^{3/2}}\right\}dz_2 \qquad (58)$$

$$+\int_z^{\ell}\phi(R,z_2)\left\{1-\frac{\frac{(z_2-z)^3}{R^3}+\frac{6(z_2-z)}{R}}{\left[\frac{(z_2-z)^2}{R^2}+4\right]^{3/2}}\right\}dz_2$$

Physically, the flux must be finite everywhere in the system so that $\phi(R,z_2)$ should be a decreasing function as the distance from the source increases. The first integral in Eq. 58 represents the contribution from the region of high neutron density while the second integral represents the contribution from the region of relatively low neutron density. Since the function $\left\{1-\frac{\left|\frac{(z-z_2)}{R}\right|^3+\frac{6|z-z_2|}{R}}{\left[\frac{|z-z_2|^2}{R^2}+4\right]^{3/2}}\right\}$ has a sharp peak at $z = z_2$ and drops off rapidly, it is reasonable to assume that the first integral has greater significance than the second. Consequently, the mean value approximation given by Eq. 48 may not be accurate for the range from 0 to z but it is a good approximation for the range z to ℓ. By referring to Eq. 48, one has

$$\frac{1}{8R}\int_z^{\ell} \phi(R,z_2)\left\{1-\frac{\frac{(z_2-z)^3}{R^3}+\frac{6(z_2-z)}{R}}{\left[\frac{(z_2-z)^2}{R^2}+4\right]^{3/2}}\right\}dz_2 \simeq \frac{\phi(R,z)}{8} \qquad (59)$$

for $\ell - z \gg R$. Hence, the Fundamental Equation can be reduced to the form

$$\begin{aligned} \frac{\phi(R,z)}{8} - \frac{1}{6\Sigma_s}\frac{\partial\phi(R,z)}{\partial r} &= \frac{S}{2}\Big[0.930\exp\left(-\frac{1.187}{R}z\right) \\ &\quad + 0.07\exp\left(-\frac{0.365}{R}z\right)\Big] \\ &\quad + \int_0^z dz_2\,\frac{\phi(R,z_2)}{8R}\Big\{0.955\exp\left[-\frac{1.187}{R}(z-z_2)\right] \\ &\quad + 0.045\exp\left[-\frac{0.447}{R}(z-z_2)\right]\Big\} \end{aligned} \qquad (60)$$

where z is much smaller than .

This equation provides the boundary condition at r = R, so that it is possible to find the solution $\phi(r,z)$ of the diffusion equation throughout the medium. If, however, z approaches ℓ, all approximations based on the diffusion theory break down because the flux gradient at the wall becomes very large due to the leakage of neutrons.

By taking the Laplace transform of Eq. 60, the integral equation is seen to reduce to an algebraic form according to the Faltung's Theorem.

$$\frac{\psi(R,s)}{8} - \frac{1}{6\Sigma_s}\frac{\partial\psi(R,s)}{\partial r} = \frac{S}{2}\left[\frac{0.930}{s + \frac{1.187}{R}} + \frac{0.07}{s + \frac{0.365}{R}}\right] + \frac{\psi(R,s)}{8R} \cdot \left[\frac{0.955}{s + \frac{1.187}{R}} + \frac{0.045}{s + \frac{0.447}{R}}\right] \tag{61}$$

where

$$\psi(R,s) = \int_0^\infty e^{-sz}\phi(R,z)\,dz \tag{62}$$

and the inverse is

$$\phi(R,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sz}\psi(R,s)\,ds \tag{63}$$

By rearranging Eq. 61, one has

$$\frac{\psi(R,s)}{4} - \frac{1}{3\Sigma_s}\frac{\partial\psi(R,s)}{\partial r} = S\,\frac{(s + \frac{0.4227}{R})(s + \frac{0.447}{R})}{(s + \frac{0.365}{R})(s^2 + \frac{0.634}{R}s + \frac{0.0506}{R})} \tag{64}$$

which provides the boundary condition in s-domain.

V. NEUTRON FLUX DISTRIBUTION IN THE MEDIUM WITH A CYLINDRICAL CAVITY

In general, the introduction of cavities in a homogeneous diffusing medium is equivalent to the reduction of the average concentration of scattering atoms per unit volume of the medium. Hence, the average scattering cross section Σ_s and the absorption cross section Σ_a decrease accordingly, and these changes are obviously dependent on the ratio of the volume of cavities to the overall volume of the system. Consequently, the closely related quantities such as the neutron mean free path, diffusion coefficient and diffusion length must, on the average, increase. Many approximations and interpretations for estimating these quantities are available in the literature. Different formulas have been cited by Paletin (12). The study of effective values of these quantities is not of a great concern in the present work. Instead, the localized variation of neutron flux in the medium and along the wall of the cavity will be discussed in detail.

Assume that the diffusion approximation is valid everywhere in the medium including boundary points. $\phi(r,z)$ must satisfy the diffusion equation in a cylindrical coordinate with axial symmetry:

$$\frac{\partial^2\phi(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial\phi(r,z)}{\partial r} + \frac{\partial^2\phi(r,z)}{\partial z^2} - k^2\phi(r,z) = 0 \qquad (65)$$

The solution of this equation, however, depends on the boundary conditions. Taking Eq. 61 as the boundary condition at $r = R$ along with others at outer boundaries, one can solve for the flux $\phi(r, z)$ throughout the medium. Note that the solution obtained this way is a good approximation for a reasonably long range of z provided that the channel is long. This restriction is rather obvious and it can be seen from the discussion on the applicability of the diffusion theory in the preceding section.

Three cases that occur quite often in reactor physics are to be considered:

A. A cylindrical cavity in an infinite diffusing medium.

B. A finite cylinder with a central cavity.

C. A limiting case for a large cavity.

In the following analysis, all neutrons are considered to be thermal. The effect of cavities on fast neutrons will be considered separately.

A. A Cylindrical Cavity in an Infinite Diffusing Medium

In cylindrical coordinates, the z-axis is taken as the axis of the cavity. Assume that an uniform plane source of infinite extent is situated at $z = 0$.

Before the cavity is introduced, the solution of the diffusion equation in an infinite homogeneous medium with an uniform isotropic-plane source has the well known form

$$\phi_1(z) = \frac{S}{2kD} \exp(-kz) \qquad (66)$$

However, when the cylindrical cavity is introduced, the flux distribution in the r-direction will no longer be constant due to the disturbance introduced by the cavity. The magnitude of this disturbance is, of course, dependent upon the relative dimension of the cavity to that of the system. For a cylindrical cavity with relatively small radius, the effect of the presence of the cavity is essentially equivalent to the introduction of a perturbation into the otherwise homogeneous system. Hence, the solution of the diffusion equation can be considered as a linear combination of an unperturbed term and a perturbed term, i.e.

$$\phi(r,z) = \phi_1(z) + \phi_2(r,z) \qquad (67)$$

where $\phi_1(z)$ is the unperturbed flux defined by Eq. 66 and $\phi_2(r,z)$ is the perturbed flux to be determined. By substitution, $\phi_2(r,z)$ must also satisfy the diffusion equation. Therefore,

$$\frac{\partial^2\phi_2(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_2(r,z)}{\partial r} + \frac{\partial^2\phi_2(r,z)}{\partial z^2} - k^2\phi_2(r,z) = 0 \qquad (68)$$

The total flux $\phi(r,z)$ must be reduced to $\phi_1(z)$ smoothly as the radius of the cavity R approaches zero. In other words,

$\phi_1(z)$ can be thought of as the solution for R = 0. Since total flux at r = R must also satisfy Eq. 60, the term $\phi(R, z)$ in Eq. 60 may be replaced by $\phi_2(R, z)$. This can be seen readily if the total flux is expressed as $\phi(r, R, z) = \phi_1(R=0, z) + \phi_2(r, R, z)$.

Furthermore, assume that the flux distribution is symmetrical with respect to the source plane at z = 0. This assumption simplifies the problem considerably and enables one to obtain the analytical solution by the method of integral transforms. In this particular case, this can be done most conveniently by the use of bilateral Laplace transforms.

1. <u>Diffusion equation and bilateral Laplace transform</u>

The bilateral Laplace transform of the function $\phi(r,z)$ is defined by

$$\Phi(r,s) = \int_{-\infty}^{\infty} f(r,s)e^{-sz}dz$$

$$= \int_{0}^{\infty} \phi(r,s)e^{-sz}dz + \int_{-\infty}^{0} \phi(r,s)e^{-sz}dz \tag{69}$$

The bilateral Laplace transform has been widely used in solving electronic problems and to some extent it has some advantages over the Fourier transform in certain problems. The theory of bilateral Laplace transform has been presented thoroughly by LePage (9) and Widder (18).

The inverse of $\psi(r,s)$ is defined as

$$\phi(r,z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(r,s) e^{sz} ds \qquad (70)$$

The first integral in Eq. 69 represents the Laplace transform of the function $\phi(r,z)$ for $z > 0$ while the second integral represents the Laplace transform of the function $\phi(r,z)$ for $z < 0$. Whether the inverse transform defined by Eq. 70 will yield the function $\phi(r,z)$ for $z > 0$ or for $z < 0$ depends upon how the contour integration is taken in evaluating Eq. 70.

Taking the bilateral Laplace transform of Eq. 68, one has

$$\frac{\partial^2 \psi_2(r,s)}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_2(r,s)}{\partial r} + \frac{\partial^2 \psi_2(r,s)}{\partial z^2} - (k^2 - s^2) \psi_2(r,s) = 0 \qquad (71)$$

The solution is obviously

$$\psi_2(r,s) = B(s) K_o(\lambda r) \qquad (72)$$

where $K_o(\lambda r)$ is the modified Bessel's function of the second kind and $\lambda = \sqrt{k^2 - s^2}$. $B(s)$ is to be determined by the boundary condition.

The inversion of Eq. 72 is, by definition,

$$\phi_2(r,z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sz} B(s) K_o(\lambda r) ds \qquad (73)$$

The inverse of $\psi_2(r,z)$ will give an unique solution if and only if there exists a region of convergence in the s-domain bounded by two vertical lines parallel to the imaginary axis, say $z = c_1$ and $z = c_2$, in which the integrand is uniformly convergent. If, in particular, $c_1 < 0 < c_2$, the bilateral Laplace transform is equivalent to the Fourier transform except with the domain axes being rotated by 90°. In considering the inverse of the transform, the upper half plane of the real axis in the w-domain for the inversion of the Fourier transform corresponds to the region to the left of the imaginary axis in s-domain in the case of bilateral Laplace transform. On the other hand, the lower half plane beneath the real axis in the w-domain corresponds to the region to the right of the imaginary axis in the s-domain. This analogy enables one to conclude that, in the case of bilateral transform, the inverse obtained by the contour integration with poles to the left of the region of convergence gives the function defined for $z > 0$ whereas the inverse obtained by the contour integration with poles to the right of the region of convergence gives the function defined in $z < 0$.

In the present problem, one is concerned only with the solution for $z > 0$. The boundary condition at $r = R$ given by Eq. 64, which was defined in terms of the unilateral Laplace transform of $\phi(r,z)$, can be thought of as the condition corresponding to the region to the left of the region of convergence. By substituting Eq. 72 into Eq. 64 one can determine

B(s), i.e. and noting that $\phi_1(z)$ corresponding to R = 0 is independent of the streaming condition

$$B(s) = \frac{\frac{S}{2}\left[(s + \frac{0.422}{R})(s + \frac{0.447}{R}\right]}{\frac{K_o(\lambda R)}{8}\left[(s^2 + \frac{0.634}{R}s + \frac{0.0506}{R^2})(s + \frac{0.365}{R})\right] + \frac{\lambda K_1(\lambda R)}{6\Sigma_s}(s + \frac{0.44}{R}}$$

so that theoretically the inverse of $\psi_2(r,s)$ can be obtained by considering only the contour to the left of the region of convergence. The result gives only the function defined for $z > 0$. To illustrate how it is done, one may consider the following approximations.

2. First approximation

Because of the complexity involved in Eq. 73, the inverse of $\psi_2(r,s)$ becomes very difficult. However, the problem can be simplified considerably by assuming that the value of $\frac{\partial\phi(R,z)}{\partial r}$ is small compared to $\phi(R,z)$ in Eq. 61. This is a reasonable assumption in the range where diffusion theory applies. Hence, Eq. 74 becomes

$$B(s) = \frac{4S(s + \frac{0.4227}{R})(s + \frac{0.447}{R})}{K_o(\lambda R)(s + \frac{0.365}{R})(s + \frac{0.094}{R})(s + \frac{0.540}{R})} \qquad (75)$$

Hence, the total flux $\phi(r,z)$ is

t $\phi_1(z)$ corresponding to R = 0 is

ning condition

$$\frac{\frac{S}{2}\left[(s + \frac{0.422}{R})(s + \frac{0.447}{R}\right]}{\frac{34}{}s + \frac{0.0506}{R^2})(s + \frac{0.365}{R}) \Big] + \frac{\lambda K_1(\lambda R)}{6\Sigma_s}(s + \frac{0.447}{R})(s + \frac{1.187}{R})(s + \frac{0.365}{R})} \qquad (74)$$

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the range where diffusion theory

becomes

$$\frac{S(s + \frac{0.4227}{R})(s + \frac{0.447}{R})}{+ \frac{0.365}{R})(s + \frac{0.094}{R})(s + \frac{0.540}{R})} \qquad (75)$$

r,z) is

$$\phi(r,z) = \phi_1(z)$$

$$+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{4s(s + \frac{0.4227}{R})(s + \frac{0.447}{R})K_0(\lambda r)e^{sz}}{K_0(\lambda R)(s + \frac{0.365}{R})(s + \frac{0.094}{R})(s + \frac{0.540}{R})} ds \tag{76}$$

where the integration is evaluated in the contour to the left of the region of convergence as shown in Fig. 8. The function $K_0(\lambda R)$ does not have any pole so long as λ is real; i.e. $|s| < (= 1/L)$ where L is the diffusion length of thermal neutrons in the medium. However, $K_0(\lambda R)$ will have two branch points at $s = \pm k$, but will have no zeros when λ is positive real or pure imaginary. The zeros of the function K_0 have been discussed by Watson (17). In the present case, $\lambda = \sqrt{k^2 - s^2}$. In addition, there are two branch points at $s = k$ and $s = -k$ at which the function K_0 is not defined. This suggests immediately the contour with branch cuts as shown in Fig. 8, by which the integrals in Eq. 76 can be evaluated. In the first integral, there are also poles at $s = -0.365/R$; $s = -0.094/R$; $s = -0.540/R$. Since there is no pole near the imaginary axis, it is always possible to find a strip, say $c_1 < s < c_2$ as shown in Fig. 8, such that the integrands in Eq. 76 is uniformly convergent. Since one is interested only in the solution corresponding to the region $z > 0$, the integration will be performed along the contour to the left of the region of convergence. Three poles of the first integrand may or may

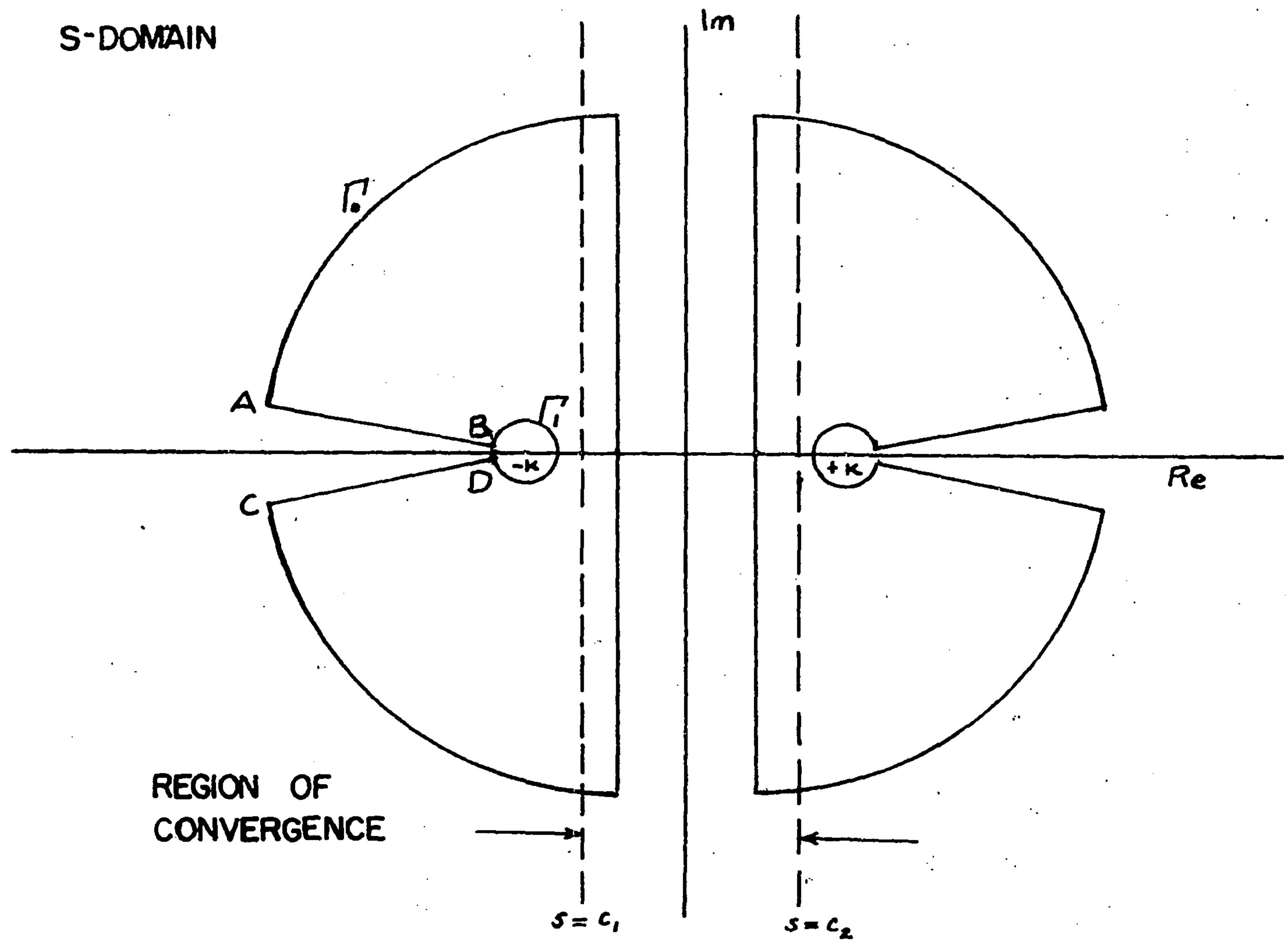


Fig. 8. The contour of integration

not be in the contour under consideration. Their relative values with respect to k determine whether or not they are located in the contour. Hence, it is possible to evaluate the integrals in Eq. 76 if the relative value of R with respect to L is known. To evaluate these integrals, three cases are to be considered.

a. Case I, $R = 0.1025\ L$

The integration in this case is along the line $s = c$ where $-0.094/R < c < 0$. Since $-0.094/R = -0.916/L > -1/L = -k$ there is only one pole $s = -0.094/R$ within the closed contour under consideration; i.e. the contour to the left of the region of convergence.

The first integral in Eq. 76 can be evaluated by the method of contour integration and calculus of residues.

The residue at $s = -0.094/R = -0.916/L$ can be evaluated readily as follows.

$$\sigma_1 = \lim_{s \to -0.094/R} \left(s + \frac{0.094}{R}\right) \cdot$$

$$\frac{4S\left(s + \frac{0.4227}{R}\right)\left(s + \frac{0.447}{R}\right)K_0(\lambda r)e^{sz}}{K_0(\lambda R)\left(s + \frac{0.365}{R}\right)\left(s + \frac{0.094}{R}\right)\left(s + \frac{0.540}{R}\right)}$$

$$= 1.06\ SK_0(0.4\ kr)\ \exp(-0.916\ kz) \tag{77}$$

Hence, the first integral in Eq. 76 becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{4S(s+\frac{0.4227}{R})(s+\frac{0.447}{R})K_o(\lambda r)e^{sz}}{K_o(\lambda R)(s+\frac{0.365}{R})(s+\frac{0.094}{R})(s+\frac{0.540}{R})}\,ds$$

$$= \sigma_1 - \frac{1}{2\pi i}\Big[\int_{\Gamma_o} 4S\frac{K_o(\lambda r)(s+\frac{0.4227}{R})(s+\frac{0.447}{R})e^{sz}}{K_o(\lambda R)(s+\frac{0.365}{R})(s+\frac{0.540}{R})}\,ds$$

$$+ \int_{\Gamma_1} + \int_{AB} + \int_{DC}\Big] \tag{78}$$

It can be easily shown that the first integral in the bracket vanishes as s approaches infinity because s assumes only negative real values in the region confined by Γ_o. The second integral around Γ_1 also vanishes when the radius of Γ_1 approaches zero. This can be shown in the following way:

By referring to Fig. 9, let $s + k = \rho e^{i\theta}$ and thus $ds = \rho\, ie^{i\theta}d\theta$. By changing the variable of integration, one has

$$\lim_{\rho \to 0}\int_{\Gamma_1}\frac{4SK_o(\lambda r)(s+\frac{0.4227}{R})(s+\frac{0.447}{R})\exp(sz)}{K_o(\lambda R)(s+\frac{0.365}{R})(s+\frac{0.540}{R})(s+\frac{0.094}{R})}\,ds$$

$$= \lim_{\rho\to 0}\exp(-kz)\int_0^{2\pi}\frac{a}{(\rho e^{i\theta} - k + \frac{0.094}{R})} \tag{79}$$

$$+ \frac{b}{(\rho e^{i\theta} - k + \frac{0.365}{R})} + \frac{c}{(\rho e^{i\theta} - k + \frac{0.540}{R})}\Big]\cdot$$

$$\exp\left[z\rho\exp(i\theta)\right] i\exp(i\theta)\,d\theta = 0$$

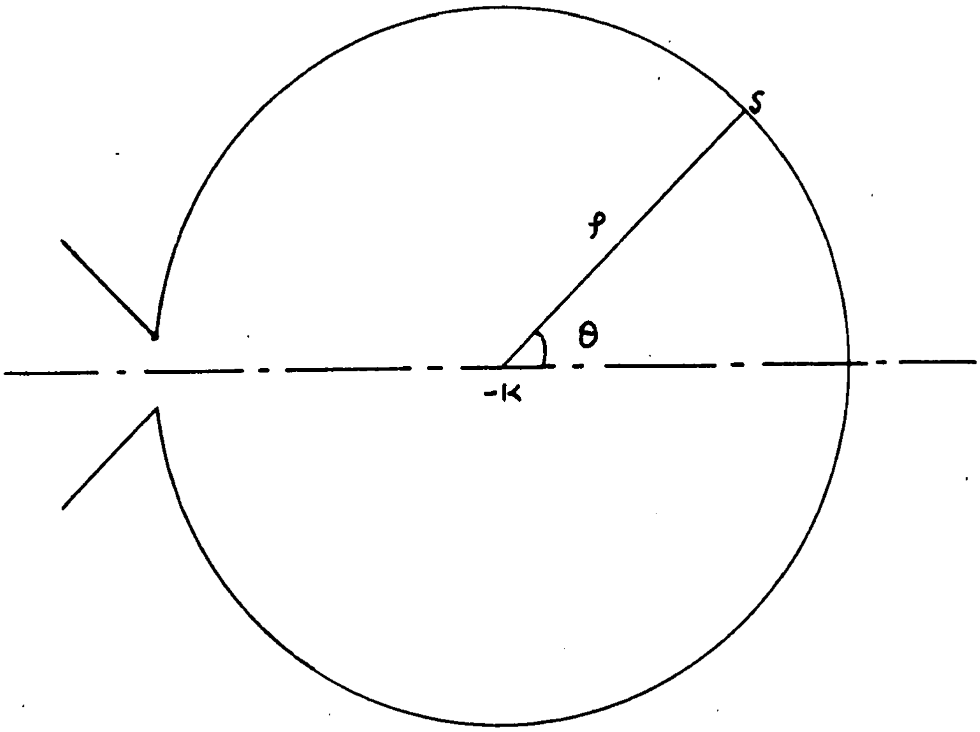


Fig. 9. Enlarged view of Γ_1

where a, b and c are constants. However, the last two line integrals do not vanish.

To evaluate the line integral along AB, let

$$u^2 e^{i\pi} = k^2 - s^2 \tag{80}$$

where u is assumed to be real and positive. The line integral $\int_{AB}$, then becomes

$$\lim_{N\to\infty} \int_N^0 \frac{2S\left(-\sqrt{k^2+u^2}+\frac{0.4227}{R}\right)\left(-\sqrt{k^2+u^2}+\frac{0.447}{R}\right)}{K_0(ue^{i\pi/2}R)\left(-\sqrt{k^2+u^2}+\frac{0.365}{R}\right)\left(-\sqrt{k^2+u^2}+\frac{0.540}{R}\right)} \cdot \frac{K_0(ue^{i\pi/2}r)\,udu}{\sqrt{k^2+u^2}\left(-\sqrt{k^2+u^2}+\frac{0.094}{R}\right)} \tag{81}$$

$$= -\int_0^\infty \frac{2S\left(-\sqrt{k^2+u^2}+\frac{0.4227}{R}\right)\left(-\sqrt{k^2+u^2}+\frac{0.447}{R}\right)}{\left(-\sqrt{k^2+u^2}+\frac{0.094}{R}\right)\left(-\sqrt{k^2+u^2}+\frac{0.365}{R}\right)\left(-\sqrt{k^2+u^2}+\frac{0.540}{R}\right)} \cdot \frac{[J_0(ur) - iY_0(ur)]e^{-\sqrt{k^2+u^2}\,z}\,udu}{[J_0(uR) - iY_0(uR)]\sqrt{k^2+u^2}}$$

where (82)

$$K_0(ure^{i\pi/2}) = -\frac{1}{2}\,i\pi H_0^{(2)}(ur) = -\frac{1}{2}\,\pi i\,[J_0(ur) - iY_0(ur)]$$

J_0, Y_0 and H_0 are Bessel's functions of the first, second and third kind respectively in Watson's notations.

Similarly, the integral along DC is just the conjugate of Eq. 80 and is equal to

$$\lim_{N\to\infty}\int_0^\infty \frac{2S(-\sqrt{k^2+u^2}+\frac{0.4227}{R})(-\sqrt{k^2+u^2}+\frac{0.447}{R})}{K_0(ue^{-i\pi/2}R)(-\sqrt{k^2+u^2}+\frac{0.365}{R})(-\sqrt{k^2+u^2}+\frac{0.540}{R})}\cdot$$

$$\frac{K_0(ure^{-i\pi/2})e^{-\sqrt{k^2+u^2}}\,udu}{\sqrt{k^2+u^2}(-\sqrt{k^2+u^2}+\frac{0.094}{R})} \tag{83}$$

$$= +\int_0^\infty \frac{2S(-\sqrt{k^2+u^2}+\frac{0.4227}{R})(-\sqrt{k^2+u^2}+\frac{0.447}{R})}{(-\sqrt{k^2+u^2}+\frac{0.365}{R})(-\sqrt{k^2+u^2}+\frac{0.540}{R})[J_0(uR)-iY_0(uR)]}\cdot$$

$$\frac{[J_0(ur)+iY_0(ur)]e^{-\sqrt{k^2+u^2}\,z}\,udu}{\sqrt{u^2+k^2}\,(-\sqrt{k^2+u^2}+\frac{0.094}{R})}$$

By substituting Eq. 79, 81, and 83 into Eq. 78, one has

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{4S(s+\frac{0.4227}{R})(s+\frac{0.447}{R})K_0(\lambda r)e^{sz}}{K_0(\lambda R)(s+\frac{0.365}{R})(s+\frac{0.094}{R})(s+\frac{0.540}{R})}$$

$$= 1.06\ SK_0(0.4kr)\exp(-0.916kz) \tag{84}$$

$$+\frac{1}{\pi}\int_0^\infty \frac{2S(-\sqrt{k^2+u^2}+\frac{0.4227}{R})(-\sqrt{k^2+u^2}+\frac{0.447}{R})}{(-\sqrt{k^2+u^2}+\frac{0.365}{R})(-\sqrt{k^2+u^2}+\frac{0.540}{R})[J_0^2(uR)+Y_0^2(uR)]}\cdot$$

$$\frac{[J_0(ur)Y_0(uR)-Y_0(ur)J_0(ur)]e^{-\sqrt{k^2+u^2}z}\,udu}{\sqrt{u^2+k^2}\,(-\sqrt{k^2+u^2}+\frac{0.094}{R})}$$

Hence, according to Eq. 76, the flux distribution in the medium can be represented by

$$\phi(r,z) = \frac{S}{2kD}\, e^{-kz} + 1.06\; SK_0(0.4kr)\exp(-0.916kz)$$

$$+ \frac{1}{\pi}\int_0^\infty \frac{2Se^{-\sqrt{k^2+u^2}\,z}\left(-\sqrt{k^2+u^2} + \frac{0.4227}{R}\right)}{\left(-\sqrt{k^2+u^2} + \frac{0.365}{R}\right)\left(-\sqrt{k^2+u^2} + \frac{0.540}{R}\right)} \cdot \tag{85}$$

$$\frac{\left(-\sqrt{k^2+u^2} + \frac{0.447}{R}\right)\left[J_0(ur)Y_0(uR)-Y_0(ur)J_0(uR)\right]}{\left(-\sqrt{k^2+u^2} + \frac{0.094}{R}\right)\left[J_0^2(uR) + Y_0^2(uR)\right]\sqrt{u^2+k^2}} \cdot$$

$$udu$$

It is remarkable to note that Eq. 85 enables one to interpret qualitatively the effect of the cylindrical channel on the flux distribution. When $r = R$, i.e. at the channel wall, the integrals in Eq. 85 vanish and thus the expression for neutron flux reduces to

$$\phi(R,z) = \frac{Se^{-kz}}{2kD} + 1.06\; SK_0(0.4kR)\exp(-0.916kz) \tag{86}$$

It is clear that the neutron flux at the channel wall is increased by an amount $1.06\; SK_0(0.4kR)\exp(-0.916kz)$ from its original value. Since R is assumed to be $0.1025\; L$, the value of $K_0(0.4kR)$ is approximately 3.45 according to values tabulated by Watson (17). The relative magnitude of this perturbed flux with respect to the unperturbed flux depends on the values of k and D according to Eq. 86. For graphite, for instance, 2kD is approximately 1/34.9. Hence, the percentage of increase for the case $R = 0.1025\; L$ is

$$\frac{\phi_2}{\phi_1} \cong \frac{3.69\ e^{-0.916kz}}{34.9\ e^{-kz}} \cong 10.6\ \%$$

for reasonable long range of z where diffusion theory is valid.

All perturbed terms in Eq. 85 are decreasing functions of r and eventually at large r far away from the channel, the total flux $\phi(r,z)$ approaches the unperturbed value. The function $\dfrac{1}{J_0^2(uR) + Y_0^2(uR)}$ is a slowly increasing function of u and approximately varies linearly with u as shown in Fig. 10. Its asymptotic expansions are given by Watson (17) in the form

$$\frac{1}{J_0^2(uR) + Y_0^2(uR)} \cong \frac{1}{\frac{2}{\pi uR}\left[1 - \frac{1}{8(uR)^2} + \cdots\right]} \cong \frac{\pi}{2}\,uR \qquad (87)$$

To estimate just how the flux varies as r increases, it is interesting to note that

$$\frac{\partial\phi(R,z)}{\partial r} = -1.06 \times 0.4kK_1(0.4kR)\exp(-0.916kz)$$

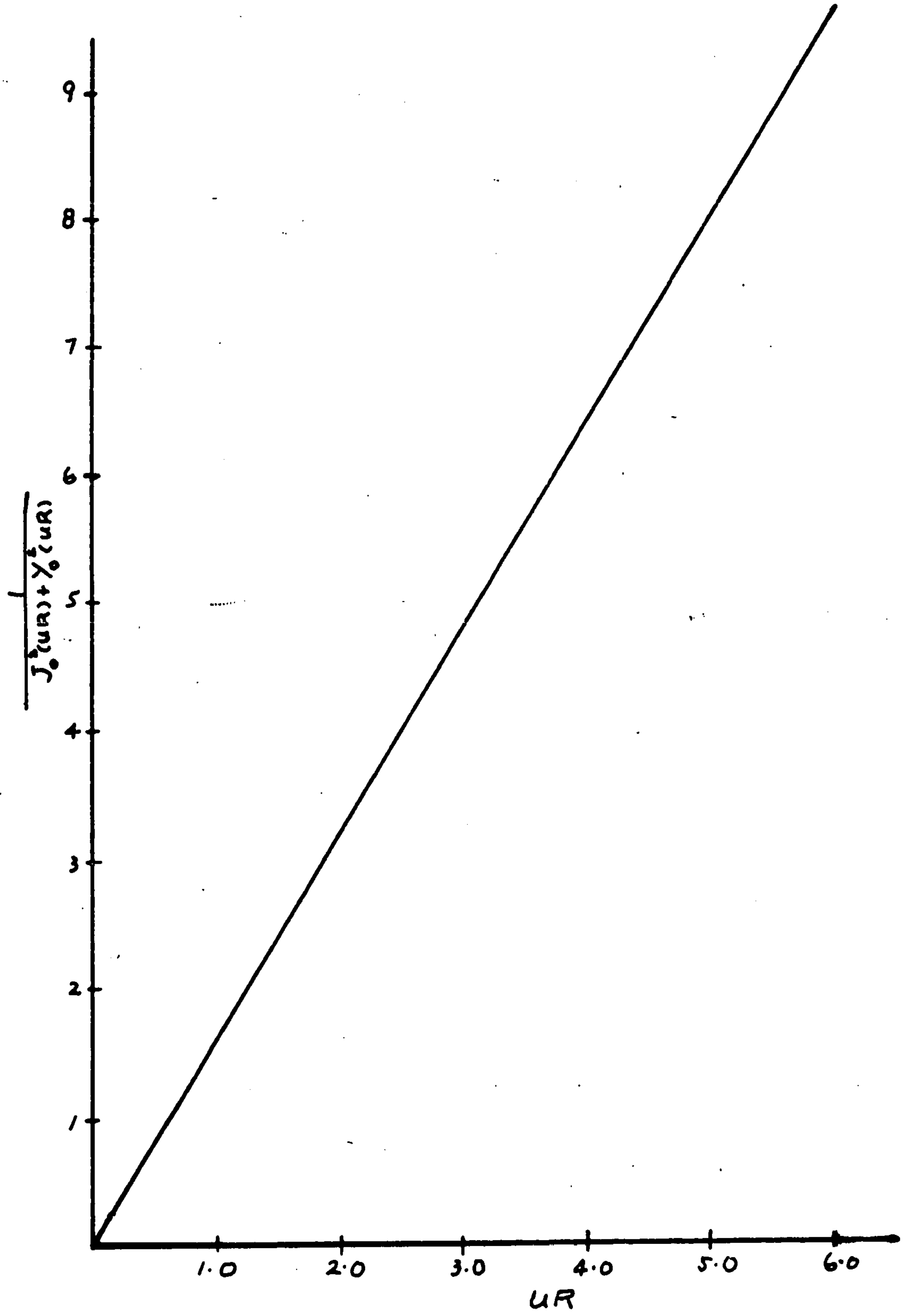
$$- \frac{1}{\pi^2 R}\int_0^\infty \frac{Se^{-\sqrt{k^2+u^2}\,z}}{\sqrt{k^2+u^2}}\left[\frac{2(-\sqrt{k^2+u^2} + \frac{0.4227}{R})}{(-\sqrt{k^2+u^2} + \frac{0.365}{R})}\cdot\right.$$

$$\left.\frac{(-\sqrt{k^2+u^2} + \frac{0.447}{R})}{(-\sqrt{k^2+u^2} + \frac{0.54}{R})(-\sqrt{k^2+u^2} + \frac{0.094}{R})}\right]\cdot$$

$$\frac{\pi}{2}\,Ru^2du \qquad (88)$$

where the identity

Fig. 10. The function $\frac{1}{J_o^2(uR) + Y_o^2(uR)}$ vs. uR


9
8
7
6
5
4
3
2
1
$\frac{1}{J_0^2(uR)+Y_0^2(uR)}$
1.0
2.0
3.0
4.0
5.0
6.0
uR

$$J_0'(uR)Y_0(uR) - Y_0'(uR)J_0(uR) = -\frac{2}{\pi uR} \qquad (89)$$

was used.

As z approaches zero, the last integral in Eq. 88 becomes infinity. For as long as $z > 0$, the integrand is a decreasing function of u. This last integral can be further approximated by

$$\frac{S}{2\pi}\int_0^\infty \frac{\exp(-\sqrt{k^2+u^2}\ z)\ u^2du}{\sqrt{k^2+u^2}}\left[\frac{2}{(-\sqrt{k^2+u^2}+\frac{0.094}{R})}\right]$$
$$= \frac{1.58\ S}{2\pi}\int_k^\infty \sqrt{\rho^2-k^2}\ e^{-\rho z}d\rho\left[\frac{2}{-\rho+\frac{0.094}{R}}\right] \qquad (90)$$

Since the function $\dfrac{e^{-\rho z}}{-\rho+\frac{0.094}{R}}$ acts very much like a Dirac δ-function in the integral and has sharp peaks at $\rho = k$, the magnitude of Eq. 90 must be very small. Hence,

$$\frac{\partial\phi(R,z)}{\partial r} \doteq -0.424\ kSK_1(0.4kR)\exp(-0.916kz) \qquad (91)$$

Note that the negative value of $\frac{\partial\phi(R,z)}{\partial r}$ along the wall means that the flux reaches a peak at the wall and decreases sharply as r increases. In other words, one would expect high neutron density in the vicinity of the channel.

b. Case II, R = 0.300 L

The flux in the medium (i.e. $r \geq R$) can be obtained by the preceding method.

$$\phi(r,z) = \frac{S}{2kD}\exp(-kz) + 1.56\, SK_0(0.95kr)\exp(-0.313kz)$$

$$+ \frac{1}{\pi}\int_0^{\infty} \frac{2Se^{-\sqrt{k^2+u^2}z}\left(-\sqrt{k^2+u^2} + \frac{0.4227}{R}\right)}{\left(-\sqrt{k^2+u^2} + \frac{0.094}{R}\right)\left(-\sqrt{k^2+u^2} + \frac{0.365}{R}\right)} \cdot \frac{\left(-\sqrt{k^2+u^2} + \frac{0.447}{R}\right)\left[J_0(ur)Y_0(uR) - Y_0(ur)J_0(uR)\right]udu}{\left(-\sqrt{k^2+u^2} + \frac{0.54}{R}\right)\left[J_0^2(uR) + Y_0^2(uR)\right]} \tag{92}$$

By comparing Eq. 92 at $r = R$ and Eq. 86, one can readily see that the flux has higher value along and near the vicinity of the channel wall for large values of R.

c. Case III, $R < 0.094\ L$

The branch point $s = -k$ in Fig. 8 is to the right of all poles in the region corresponding to the solution of $z > 0$. Hence,

$$\phi(r,z) = \phi_1(z) + \int_0^{\infty} \frac{2Se^{-\sqrt{k^2+u^2}z}\left(-\sqrt{k^2+u^2} + \frac{0.4227}{R}\right)}{\left(-\sqrt{k^2+u^2} + \frac{0.365}{R}\right)\left(-\sqrt{k^2+u^2} + \frac{0.54}{R}\right)} \cdot \frac{\left(-\sqrt{k^2+u^2} + \frac{0.447}{R}\right)\left[J_0(ur)Y_0(uR) - Y_0(ur)J_0(uR)\right]udu}{\left(-\sqrt{k^2+u^2} + \frac{0.094}{R}\right)\sqrt{u^2+k^2}\left[J_0^2(uR) + Y_0^2(uR)\right]} \tag{93}$$

When $r = R$, Eq. 93 becomes $\phi(R,z) = \phi_1(z)$ = unperturbed flux. It is clear that the flux does not change appreciably if the radius of the channel is very small. The perturbed flux for small values of R can not be determined by the preceding method. A second approximation will be given in the following section.

3. Second approximation

All the preceding approximations are based on the assumption that the value of $\frac{\partial \phi(R,z)}{\partial r}$ is small compared to $\phi(R,z)$. If $\frac{\partial \phi(R,z)}{\partial r}$ is not neglected from Eq. 60, the substitution of Eq. 72 into Eq. 64 gives

$$B(s) = \frac{\frac{S}{2}\, h(s)}{\left[\frac{K_0(\lambda R)}{8}\, f(s) + \frac{\lambda K_1(\lambda R)}{6\Sigma_s}\, g(s)\right](s + \frac{0.365}{R})} \tag{94}$$

and hence,

$$\phi(r,s) = \phi_1(s) + \frac{\frac{S}{2}\, h(s) K_0(\lambda r)}{\left[\frac{K_0(\lambda R)}{8}\, f(s) + \frac{\lambda K_1(\lambda R)}{6\Sigma_s}\, g(s)\right](s + \frac{0.365}{R})} \tag{95}$$

where

$$f(s) = (s + \frac{0.540}{R})(s + \frac{0.094}{R}) \tag{96}$$

$$g(s) = (s + \frac{0.447}{R})(s + \frac{1.187}{R}) \tag{97}$$

$$h(s) = (s + \frac{0.422}{R})(s + \frac{0.447}{R}) \tag{98}$$

The transcendental function of the form

$$\frac{K_0(\lambda R) f(s)}{8} + \frac{\lambda K_1(\lambda R) g(s)}{6\Sigma_s} \tag{99}$$

will not have zeros for real $\lambda > 0$ but it will have branch points at $s = \pm k$. The discussion of this type of functions has been given by Erdelyi (7).

Again, for the solution corresponding to $z > 0$, the left contour in Fig. 8 will be used. If $R < 0.365\ L$, there is no pole within the contour under consideration. The inverse of Eq. 95 becomes

$$\phi(r,z) = \frac{S}{2kD}\, e^{-kz} - \frac{1}{\pi}\int_0^{\infty} h(-\sqrt{k^2+u^2})\, e^{-\sqrt{k^2+u^2}\,z} \ .$$

$$\frac{\frac{S}{2}\ J_0(ur)\left[\frac{f(-\sqrt{k^2+u^2})}{8}Y_0(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}Y_1(uR)\right] - Y_0(ur)\left[\frac{f(-\sqrt{}}{}\right.}{\left[\frac{f(-\sqrt{k^2+u^2})}{8}J_0(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}J_1(uR)\right]^2 + \left[\frac{f(-\sqrt{k^2+u^2})}{8}Y_0(u\right.}$$

$$\frac{K_0(\lambda r)\,u\,du}{\left(-\sqrt{k^2+u^2} + \frac{0.365}{R}\right)\sqrt{k^2+u^2}}$$

This integral, in principle, can be evaluated by the numerical method. For practical purpose, the results obtained by the first approximation is sufficiently good estimation for qualitative results so long as z is far away from the end of the medium.

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lution corresponding to $z > 0$, the left
be used. If $R < 0.365\ L$, there is no
r under consideration. The inverse of

$$\int_0^\infty h(-\sqrt{k^2+u^2})\,e^{-\sqrt{k^2+u^2}\,z} \; .$$

$$\frac{\overline{{}^2+u^2)}Y_o(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}Y_1(uR)\Big] - Y_o(ur)\left[\frac{f(-\sqrt{k^2+u^2})}{8}J_o(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}J_1(uR)\right]}{{}_o(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}J_1(uR)\Big]^2 + \left[\frac{f(-\sqrt{k^2+u^2})}{8}Y_o(uR) + \frac{\lambda g(-\sqrt{k^2+u^2})}{6\Sigma_s}Y_1(uR)\right]^2}$$

$$\overline{}\quad (100)$$

$$)\ \sqrt{k^2+u^2}$$

ciple, can be evaluated by the numer-
tical purpose, the results obtained by
n is sufficiently good estimation for
long as z is far away from the end

4. Neutron flux in the cavity

Assume there are no scattering and absorption processes in air. It has been discussed previously that the neutron flux in the absence of scattering atoms must satisfy the Laplace equation.

Thus, at any given z_o, the flux satisfies

$$\frac{\partial^2 \phi(r, z_o)}{\partial r^2} + \frac{1}{r} \frac{\partial \phi(r, z_o)}{\partial r} = 0 \qquad (101)$$

Since $\phi(r, z_o)$ is everywhere continuous and bounded in the circular region bounded by $r = R$ at $z = z_o$, $\phi(r, z_o)$ must be a harmonic function in that region. By the Maximum Principle for harmonic functions, the value of $\phi(r, z)$ in the region must exceed its maximum value on the boundary, i.e. $\phi(R, z_o)$, and never assumes its minimum value in the region unless $\phi(R, z_o)$ is constant. Because of the axial symmetry of the flux about the z-axis, one is led to the conclusion that $\phi(r, z_o)$ must be constant or else one would expect a maximum or a minimum on z-axis.

Therefore, one can write

$$\phi(r, z) = \phi(R, z) \quad \text{for} \quad r \leq R$$

B. Cylinder with Finite Radius

Consider a finite cylindrical column, whose dimension in axial direction is very large compared to its dimension in radial direction, and containing a cylindrical co-axial cavity. The radii of the column and the cavity are designated by a_1 and R respectively. If $a_1/R >> 1$, the Fundamental Equation discussed previously is assumed to be valid. Again, the neutron flux is assumed to satisfy the diffusion equation everywhere in the column except in the cavity.

A plane source is assumed to be located at z = 0. Without losing generality, the plane source in the form of a circular plate is assumed to satisfy the relation

$$S(r) = S_0 J_0(\mu a) \tag{102}$$

where S_0 is a constant and μ is the solution of the transcendental equation $J_0(\mu a) = 0$ with "a" being the extrapolated radius of the system.

Eq. 102 is satisfied in many practical cases; for instance, the source may very well be the core of a slab reactor or a fission plate. It is easy to show that the neutron flux in a homogeneous cylindrical system without any cavity is

$$\phi_1(r,z) = \frac{S_0}{2\sqrt{k^2+\mu^2}\ D} J_0(\mu r) \exp(-\sqrt{k^2+\mu^2}\ z) \tag{103}$$

where $\phi_1(a,z) = 0$ and length of the column is large.

If $a/R >> 1$, the solution of the diffusion equation can be considered again as a linear combination of the unperturbed flux and the perturbed flux designated by ϕ_1 and ϕ_2 respectively, where ϕ_1 is defined by Eq. 103. Assume that the source strength is not affected by the presence of the cavity.

Taking bilateral Laplace transform of the diffusion equation and using the same argument as the case of infinite medium, one has

$$\frac{\partial^2 \psi_2(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_2(r,s)}{\partial r} + (s^2 - k^2)\psi_2(r,s) = 0 \qquad (104)$$

so that

$$\psi_2(r,s) = A(s)J_0(\sigma r) + B(s)Y_0(\sigma r) \qquad (105)$$

where $\sigma = \sqrt{s^2 - k^2}$ and $J_0(\sigma r)$ and $Y_0(\sigma r)$ are Bessel's functions of the first and second kind respectively. By imposing the boundary condition $\psi_2(a,s) = 0$, one obtains

$$\psi_2(r,s) = B(s)\left[\frac{Y_0(\sigma r)J_0(\sigma a) - J_0(\sigma r)Y_0(\sigma a)}{J_0(\sigma a)}\right] \qquad (106)$$

Again, the value of $\frac{\partial \phi(R,z)}{\partial r}$ is assumed to be small compared to $\phi(R,z)$ as what would be expected when $a/R >> 1$. Since

$$\psi(r,s) = \psi_1(r,s) + \psi_2(r,s) \qquad (107)$$

the substitution of Eq. 107 into Eq. 64 gives

$$B(s) = \frac{4S_0\left[\frac{0.930}{s + \frac{1.187}{R}} + \frac{0.07}{s + \frac{0.365}{R}}\right] J_0(\sigma a)}{[Y_0(\sigma R)J_0(\sigma a) - J_0(\sigma R)Y_0(\sigma a)]\left[1 - \frac{1}{R}\left(\frac{0.955}{s + \frac{1.187}{R}} + \frac{0.045}{s + \frac{0.447}{R}}\right)\right]}$$

$$= \frac{4S_o(s + \frac{0.4227}{R})(s + \frac{0.447}{R})J_o(\sigma a)}{[Y_o(\sigma R)J_o(\sigma a) - J_o(\sigma R)Y_o(\sigma a)](s + \frac{0.365}{R})(s + \frac{0.094}{R})} \cdot$$

$$\frac{1}{(s + \frac{0.540}{R})} \qquad (107)$$

Note that $\phi_1(r,z)$ corresponding to the solution for $R = 0$ is again independent of the streaming condition. Hence, the solution becomes

$$\phi(r,z) = \phi_1(r,z) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{4S_o(s + \frac{0.4227}{R})}{[Y_o(\sigma R)J_o(\sigma a) - J_o(\sigma R)Y_o(\sigma a)]}$$

$$\frac{(s + \frac{0.447}{R})e^{sz}[Y_o(\sigma R)J_o(\sigma a) - J_o(\sigma R)Y_o(\sigma a)]ds}{(s + \frac{0.365}{R})(s + \frac{0.094}{R})(s + \frac{0.54}{R})} \qquad (108)$$

These line integrals can be evaluated by transforming them into a closed contour as shown in Fig. 11. Since one is only interested in the solution for the case $z > 0$, only the contour to the left of the imaginary axis will be considered. It can be readily shown that

$$\frac{1}{2}\pi[Y_o(|\sigma R|e^{\frac{i\pi}{2}})J_o(|\sigma a|e^{\frac{i\pi}{2}}) - J_o(|\sigma R|e^{\frac{i\pi}{2}})Y_o(|\sigma a|e^{\frac{i\pi}{2}})]$$

$$= [I_o(\sigma a)K_o(\sigma R) - K_o(\sigma a)I_o(\sigma R)] \qquad (109)$$

where I_o and K_o are modified Bessel's functions of the first

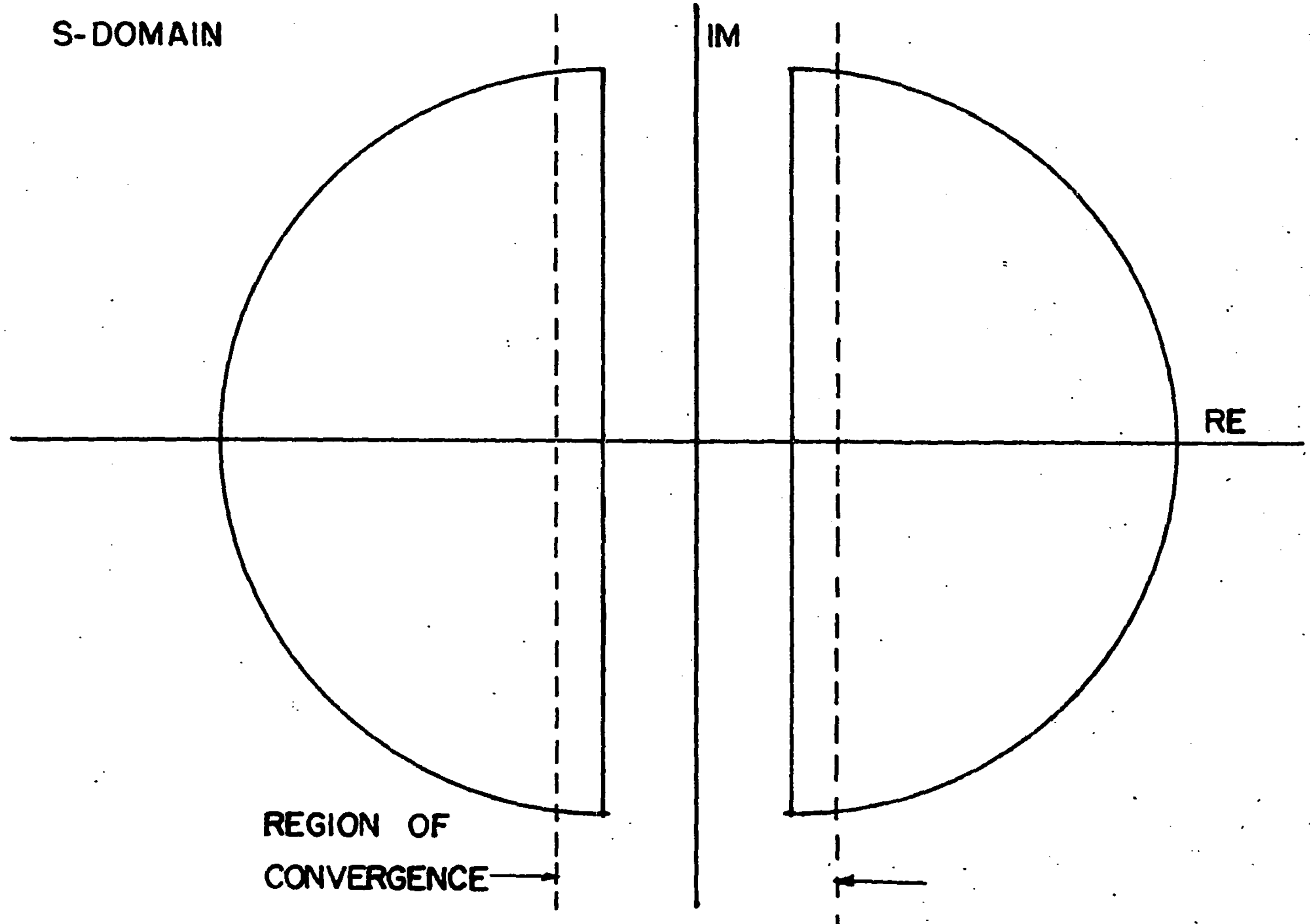

Fig. 11. The contour of integration

and second kind respectively. Therefore, it is possible to define a strip in the vicinity of the imaginary axis in s-domain such that the integrands in Eq. 108 are uniformly convergent in this region. The previous argument is, therefore, applicable in this case.

In the first integral, there are three poles in addition to a sequence of poles at $s = -\alpha_1, -\alpha_2, \cdots, -\alpha_i$ where $\sqrt{\alpha_i^2-k^2}$ is a root of the transcendental equation

$$Y_0(\sigma R)J_0(\sigma a) - J_0(\sigma R)Y_0(\sigma a) = 0 \tag{110}$$

and α_i is positive and real.

Let

$$\beta^2 = (-\frac{0.365}{R})^2 - k^2 \tag{111}$$

$$\gamma^2 = (-\frac{0.094}{R})^2 - k^2 \tag{112}$$

$$\omega^2 = (-\frac{0.540}{R})^2 - k^2 \tag{113}$$

The residues corresponding to $s = -0.365/R$; $-0.094/R$ and $-0.54/R$ can be readily obtained and their values are

$$\rho_\beta = -\frac{0.399 S_0[Y_0(\beta r)J_0(\beta a)-J_0(\beta r)Y_0(\beta a)]}{[Y_0(\beta R)J_0(\beta a)-J_0(\beta R)Y_0(\beta a)]}\exp(-\frac{0.365}{R}z) \tag{114}$$

$$\rho_\gamma = +\frac{3.84\ S_0[Y_0(\gamma r)J_0(\gamma a)-J_0(\gamma r)Y_0(\gamma a)]}{[Y_0(\gamma R)J_0(\gamma a)-J_0(\gamma R)Y_0(\gamma a)]}\exp(-\frac{0.094}{R}z) \tag{115}$$

$$\rho_\omega = +\frac{0.655 S_0[Y_0(\omega r)J_0(\omega a)-J_0(\omega r)Y_0(\omega a)]}{[Y_0(\omega R)J_0(\omega a)-J_0(\omega R)Y_0(\omega a)]}\exp(-\frac{0.540}{R}z) \tag{116}$$

respectively.

To evaluate the residue at $s = -\alpha_1$, or $\sigma = \sqrt{\alpha_1^2 - k^2}$, one may write

$$\rho_1 = \lim_{s \to \alpha_1} \frac{4S_0(s + \alpha_1)(s + \frac{0.4227}{R})(s + \frac{0.447}{R})}{[Y_0(\sigma R)J_0(\sigma a) - J_0(\sigma R)Y_0(\sigma a)]} \cdot \frac{e^{sz}[Y_0(\sigma r)J_0(\sigma a) - J_0(\sigma r)Y_0(\sigma a)]}{(s + \frac{0.365}{R})(s + \frac{0.094}{R})(s + \frac{0.540}{R})} \tag{117}$$

By applying La Hospital's rule, and using the identity

$$J_0'(\sigma a)Y_0(\sigma R) - J_0(\sigma R)Y_0'(\sigma a) = \frac{1}{\sigma a}\frac{J_0(\sigma R)}{J_0(\sigma a)} \tag{118}$$

given by Sneddon (15), Eq. 117 becomes

$$\rho_1 = \frac{4S_0'(\alpha_1 + \frac{0.4227}{R})(\alpha_1 + \frac{0.447}{R})\exp(-\alpha_1 z)}{\dfrac{\alpha_1[J_0^2(\sqrt{\alpha_1^2-k^2}\, a) - J_0^2(\sqrt{\alpha_1^2-k^2}\, R)]}{(\alpha_1^2 - k^2)J_0(\sqrt{\alpha_1^2-k^2}\, R)J_0(\sqrt{\alpha_1^2-k^2}\, a)}} \cdot \frac{[Y_0(\sqrt{\alpha_1^2-k^2}\, r)J_0(\sqrt{\alpha_1^2-k^2}\, R) - J_0(\sqrt{\alpha_1^2-k^2}\, r)Y_0(\sqrt{\alpha_1^2-k^2}\, R)]}{(\alpha_1 + \frac{0.365}{R})(\alpha_1 + \frac{0.094}{R})(\alpha_1 + \frac{0.54}{R})} \tag{119}$$

Hence, the inverse defined by Eq. 108 is just the summation of all residues in the region under consideration.

$$\phi(r,z) = \frac{S_o J_o(\mu a) e^{-\sqrt{\mu^2+k^2}z}}{2D\sqrt{k^2+\mu^2}}$$

$$+ \frac{3.84 S_o e^{-0.094z/R} \left[Y_o(\gamma r) J_o(\gamma a) - J_o(\gamma r) Y_o(\gamma a) \right]}{\left[Y_o(\gamma R) J_o(\gamma a) - J_o(\gamma R) Y_o(\gamma a) \right]}$$

$$+ \frac{0.655 S_o e^{-0.54z/R} \left[Y_o(\omega r) J_o(\omega a) - J_o(\omega r) Y_o(\omega a) \right]}{\left[Y_o(\omega R) J_o(\omega a) - J_o(\omega R) Y_o(\omega a) \right]}$$

$$- \frac{0.399 S_o \left[Y_o(\beta r) J_o(\beta a) - J_o(\beta r) Y_o(\beta a) \right] e^{-0.365z/R}}{\left[Y_o(\beta R) J_o(\beta a) - J_o(\beta R) Y_o(\beta a) \right]}$$

$$+ \sum_{i=1}^{\infty} \frac{\sqrt{(\alpha_i^2-k^2)}\, e^{-\alpha_i z} J_o(\sqrt{\alpha_i^2-k^2}\, a) J_o(\sqrt{\alpha_i^2-k^2}\, R)}{\alpha_i \left[J_o^2(\sqrt{\alpha_i^2-k^2}\, a) - J_o^2(\sqrt{\alpha_i^2-k^2}\, R) \right]} \cdot \tag{120}$$

$$\frac{4S(\alpha_i + \frac{0.4227}{R})(\alpha_i + \frac{0.447}{R})}{(\alpha_i + \frac{0.365}{R})(\alpha_i + \frac{0.094}{R})(\alpha_i + \frac{0.540}{R})} \cdot$$

$$\left[Y_o(\sqrt{\alpha_i^2-k^2}\, r) J_o(\sqrt{\alpha_i^2-k^2}\, a) - J_o(\sqrt{\alpha_i^2-k^2}\, r) \cdot Y_o(\sqrt{\alpha_i^2-k^2}\, a) \right]$$

This solution is much more complicated than that of the case of infinite medium due to the presence of the harmonic terms. Eq. 120 clearly indicates that the neutron flux can not be related to other parameters in a simple manner. The first term represents the unperturbed flux. Three following terms represent the contribution of streaming neutrons across the channel. These terms decrease rapidly as r increases. The rate of decrease depends on the ratio a/R. The contribution of terms under the summation sign also depends on the ratio a/R. For large a/R, the roots $\{\alpha_i\}$ of Eq. 110 are small so that one would expect an appreciable contribution from these harmonic terms. On the other hand, if a/R is not too large, the roots $\{\alpha_i\}$ of Eq. 110 are expected to be of appreciable magnitude so that all these terms in the summation sign reduce rapidly to zero at some distance away from the source due to the exponential term $e^{-\alpha_i z}$. To illustrate the dependence of α_i on a/R, the first five roots of Eq. 110 given by Carslaw and Jaeger (5) is tabulated as follows:

a/R	α_1	α_2	α_3	α_4	α_5
1.5	6.2702	12.5598	18.8451	25.1294	31.4133
2.0	3.1230	6.2734	9.4182	12.5614	15.7040
3.0	1.5485	3.1291	4.7038	6.2767	7.8487
4.0	1.0244	2.0809	3.1322	4.1816	5.2301

It is obvious that all terms in the summation sign in Eq. 120 can be neglected for a/R $\leq$ 5 at a reasonable distance away from the source. Then, the flux distribution reduces to the form

$$\phi(r,z) \cong \frac{S_o J_o(\mu r)}{2D\sqrt{\mu^2+k^2}} \exp(-\sqrt{k^2+\mu^2}\, z)$$

$$+\frac{3.84\ S_o e^{-0.094z/R}\left[Y_o(\gamma r)J_o(\gamma a)-J_o(\gamma r)Y_o(\gamma a)\right]}{\left[Y_o(\gamma R)J_o(\gamma a)-J_o(\gamma R)Y_o(\gamma a)\right]}$$

$$+\frac{0.655\ S_o e^{-0.54z/R}\left[Y_o(\omega r)J_o(\omega a)-J_o(\omega r)Y_o(\omega a)\right]}{\left[Y_o(\omega R)J_o(\omega a)-J_o(\omega R)Y_o(\omega a)\right]} \qquad (121)$$

$$-\frac{0.399\ S_o e^{-0.365z/R}\left[Y_o(\beta r)J_o(\beta a)-J_o(\beta r)Y_o(\beta a)\right]}{\left[Y_o(\beta R)J_o(\beta a)-J_o(\beta R)Y_o(\beta a)\right]}$$

On the other hand, if a/R is large, say $a/R \geq 10$, the contribution of the harmonic terms will become significant.

C. A Limiting Case of a Large Cavity

Consider that there exists an air gap in a diffusing medium of infinite extent. This is the case in which the diameter of the cavity becomes comparable to that of the diffusing medium. In this case, the diffusion equation is essentially one dimensional.

Assume that the air gap with width ℓ_2 is sandwiched between Region I and Region III as shown in Fig. 12. It is further assumed that the diffusion equation is valid in Region I and Region III, whereas the neutron flux distribution satisfies the continuity equation defined by

$$\bar{\Omega} \cdot \text{grad}\ \phi = 0$$

in region II. For one dimensional case, this becomes

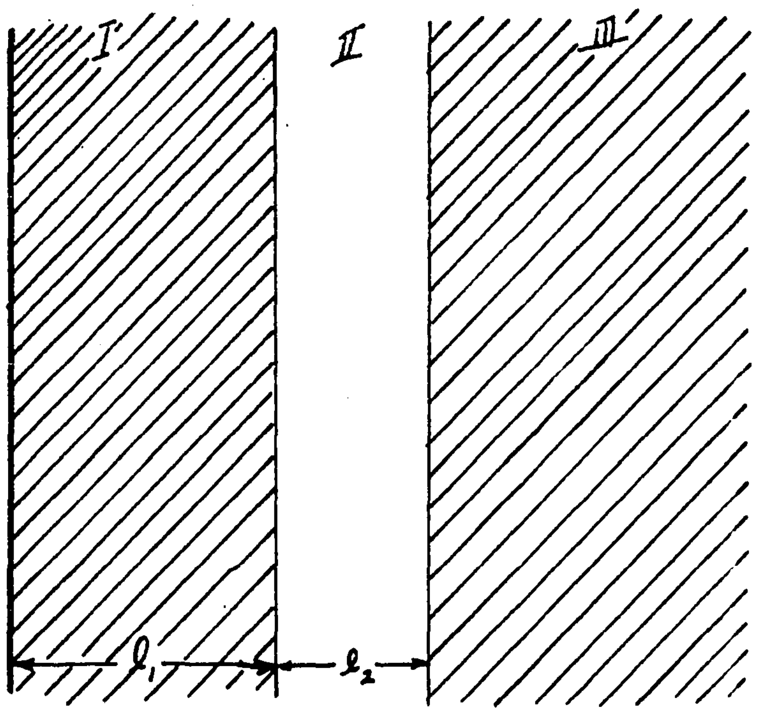

Fig. 12. A limiting case

$$\frac{\partial \phi}{\partial z} = 0 \tag{122}$$

which means

$$\phi(z) = \text{constant} \tag{123}$$

in Region II.

In region I, the solution of a one dimensional diffusion equation is of the form

$$\phi(z) = A \cosh kz + B \sinh kz \tag{124}$$

In region III, the solution is of the form

$$\phi_3(z) = Ce^{-kz} \tag{125}$$

if the medium is extended to infinity.

To determine parameters A, B and C, the following boundary conditions are seen to apply.

1. $-D \left.\frac{d\phi_1}{dz}\right|_{z=0} = \frac{S}{2}$, where S is again the source strength.
2. $\left.\frac{d\phi_1}{dz}\right|_{z=\ell_1} = \left.\frac{d\phi_3}{dz}\right|_{z=\ell_1+\ell_2} = 0$
3. $\phi_1(\ell_1) = \phi_3(\ell_1+\ell_2)$ which is the consequence of Eq. 123.

From boundary conditions 1, 2, and 3, one has

$$B = -\frac{S}{2kD} \tag{126}$$

$$A = \frac{S \cosh k\ell_1}{2kD \sinh k\ell_1} \tag{127}$$

$$C = \frac{S \exp[k(\ell_1 + \ell_2)]}{2kD \sinh k\ {}_1} \tag{128}$$

Hence, the expressions of the flux become

$$\phi_1(z) = \frac{S}{2kD}\left[\frac{\cosh k\ell_1}{\sinh k\ell_1} \cosh kz - \sinh kz\right] \tag{129}$$

$$\phi_3(z) = \frac{S \exp\{-k[z - (\ell_1 + \ell_2)]\}}{2kD \sinh k\ {}_1} \tag{130}$$

Since $\cosh kz \geq \sinh kz$ for all z, the value of $\left(\frac{\cosh k\ell_1}{\sinh k\ell_1} \cosh kz - \sinh kz\right)$ is greater than $\exp(-kz)$, which is the solution for the homogeneous case. The effect of the air gap then is to raise the neutron flux in the medium by reducing the flux gradient in the vicinity of the gap.

VI. INFINITE MEDIUM WITH A SPHERICAL CAVITY

Consider an infinite diffusion medium containing a spherical cavity with radius R. Assume that there exists an uniform plane source at a distance "a" from the center of the cavity and $a/R \gg 1$ as shown in Fig. 13. Since there is no neutron created or absorbed in D_2, the spherical cavity acts essentially as a neutron trap in the medium. Neutrons that enter the region D_2 may be scattered back and forth by the cavity wall. Hence, one may expect an increase of neutron density in this region. It is further assumed that the diffusion theory is valid everywhere in the medium including boundary points and that the flux must satisfy the Laplace equation discussed previously. Physically, the flux must be everywhere continuous in $D_2 + B$. It follows that the flux ϕ is harmonic in D_2. The problem of finding ϕ in D_2 becomes the well known Dirichlet problem. Once the value of the flux on the boundary is specified, the determination of ϕ in D_2 becomes rather simple.

The most important property of a harmonic function is that it satisfies the mean value property; i.e.

$$\phi(P) = \frac{1}{4\pi R^2} \iint_B \phi \, dA \tag{131}$$

where P is the point at the center of the spherical cavity.

Eq. 131 immediately suggests that the effect of the

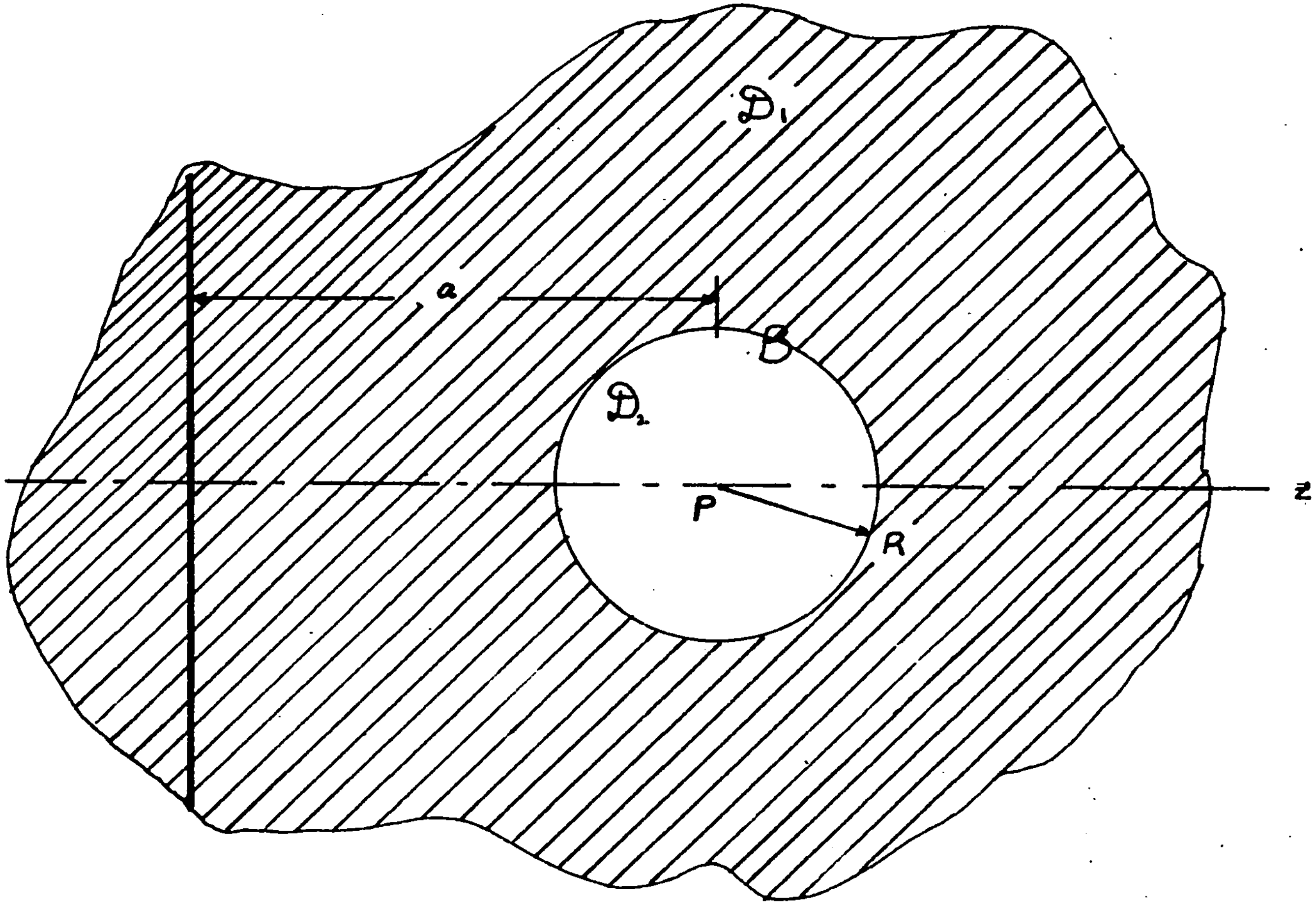

Fig. 13. An infinite medium containing a spherical cavity

spherical cavity can be considered as if an additional neutron source were added to the otherwise homogeneous system at the point P. The effect, therefore, is equivalent to the introduction of a perturbation into the homogeneous system. It is possible to determine the flux distribution in this perturbed system.

A. Perturbation Approximation

The unperturbed flux in a homogeneous system with an uniform plane source can be represented by

$$\phi_1 = \frac{S}{2kD} \exp(-kz) \tag{132}$$

For mathematical simplification in solving for the flux, it is most convenient to consider a spherical coordinate system with the origin placed at the point P. By referring to Fig. 12, the flux distribution must have axial symmetry with respect to the z-axis. If angle θ is the angle between any r in the spherical coordinate and z-axis, Eq. 132 may be expressed in an alternative form in terms of r and θ in the spherical coordinate; i.e.

$$\phi_1(r,\theta) = \frac{S}{2kD} \exp\left[-k(r \cos \theta + a)\right] \tag{133}$$

It can be shown readily by direct substitution that Eq. 133 satisfies the diffusion equation expressed in the spherical coordinate:

$$\frac{1}{r^2}\left[r\,\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial\phi}{\partial\theta}) \right] - k^2\phi = 0 \qquad (134)$$

Replacing the cavity by a fictitious source S_2, one can determine the perturbed flux caused by the presence of the spherical cavity. Since the net number of neutrons cross the boundary B is constant and the flux has the mean value property defined by Eq. 131, the fictitious source S_2 can be thought of as being a constant point source located at point P. Furthermore, the increase in the localized region in and around D_2 arises from the fact that most neutrons entering the cavity are trapped by the back and forth scattering on the boundary. It is, therefore, reasonable to assume that S_2 is isotropic if the scattering on the boundary is assumed to be isotropic. Hence, the perturbed flux determined by S_2 must be independent of the angle θ. The total flux must have the form

$$\phi(r,\theta) = \phi_1(r,\theta) + \phi_2(r) \qquad (135)$$

This total neutron flux $\phi(r,\theta)$ satisfies the Laplace equation $\nabla^2\phi = 0$ in D_2. Suppose there exists a function u which is continuous and differentiable in $B + D_2$. By the first Green's identity, one has

$$\iiint_{D_2} u\,\nabla^2\phi\;dV + \iiint_{D_2}(u_x\phi_x + \phi_y u_y)\,dV = \iint_A u\,\frac{\partial\phi}{\partial r}\,dA \qquad (136)$$

where A is the surface area of the sphere.

Let u = 1, then Eq. 136 becomes

$$\iint_A \frac{\partial \phi}{\partial r}\, dA = 0 \tag{137}$$

This condition can be used to determine the value of the fictitious source S_2.

Substituting Eq. 135 into Eq. 134, one has

$$\frac{\partial^2}{\partial r^2}(r\phi_2) - k^2(r\phi_2) = 0 \tag{138}$$

If the boundary condition is such that

$$\lim_{r \to R} -D\left(\frac{\partial \phi}{\partial r}\right) = \frac{S_2}{4\pi R^2} \tag{159}$$

the solution of Eq. 138 becomes

$$\phi_2(r) = \frac{S_2 \exp[-k(r - R)]}{4\pi Dr[1 + kR]} \text{ for } r \geq R \tag{140}$$

Hence, the total flux of the system becomes

$$\phi(r,z) = \phi_1(r,\theta) + \phi_2(r) \tag{141}$$

$$= \frac{S}{2kD} \exp[-k(r \cos\theta + a)] + \frac{S_2 \exp[-k(r-R)]}{4\pi Dr[1 + kR]}$$

for $r \geq R$.

According to Eq. 137, $\phi(r,z)$ must also satisfy

$$\iint_A \frac{\partial \phi}{\partial r}\, dA = \int_0^{\pi} \frac{\partial \phi_1(R,\theta)}{\partial r} 2\pi R^2 \sin\theta \, d\theta + \int_0^{\pi} \frac{\partial \phi_2(R)}{\partial r} 2\pi R^2 \sin\theta \, d\theta$$

$$= 0 \tag{142}$$

or

$$\int_0^{\pi} -2\pi \, \frac{S \exp[-k(R \cos \theta + a)]}{2D} \cos \theta R^2 \sin \theta \, d\theta$$

$$+ \int_0^{\pi} \frac{S_2}{4\pi D(1 + kR)} \{-kR-1\} \frac{2\pi R^2 \sin \theta \, d\theta}{R^2} = 0 \tag{143}$$

Hence, by simple integration, one has

$$\frac{S \exp(-ka)(kR \cosh kR - \sinh kR)}{Dk^2} - \frac{S_2}{D} = 0$$

and

$$S_2 = \frac{S}{k^2} \exp(-ka)(kR \cosh kR - \sinh kR) \tag{144}$$

Then, the flux for $r \geq R$ becomes

$$\phi(r,\theta) = \frac{S}{2kD} \exp[-k(r \cos \theta + a)]$$

$$+ \frac{S}{k^2}(kR \cosh kR - \sinh kR) \exp[-k(r + a - R)] \tag{145}$$

for $r \geq R$. Note that the perturbed term vanishes when R approaches zero and the total flux reduced to the unperturbed flux. When $R > 0$, the perturbed term begins to contribute. The transcendental function $\{kR \cosh kR - \sinh kR\}$ is an increasing function of R so that the contribution of the perturbed term depends on the value kR or R/L. If R/L becomes greater than the order of unity, this perturbed term becomes large compared to the first term and eventually the perturbation calculation will breakdown.

B. Flux Distribution in the Cavity

The flux distribution in D_2 can be determined when $\phi(R,\theta)$ is known. From Eq. 145, one has

$$\begin{aligned}\phi(R,\theta) = {} & \frac{S}{2kD} \exp\left[-k(r \cos\theta + a)\right] \\ & + \frac{S}{k^2}(kR \cosh kR - \sinh kR)\exp(-ka)\end{aligned} \tag{146}$$

In the spherical coordinate, the flux in D_2 satisfies the Laplace equation of the form

$$r \frac{\partial^2(r\phi)}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial\phi}{\partial\theta}) = 0 \tag{147}$$

for $r < R$.

Assume that the function $\phi(r,\theta)$ is separable in r and θ. Let $\phi(r,\theta) = G(r)F(\theta)$. By substitution, Eq. 147 becomes

$$r \frac{\partial^2}{\partial r}(rG) = \lambda G \tag{148}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\frac{1 - \cos^2\theta}{\sin\theta}\frac{dF}{d\theta}) + \lambda F = 0 \tag{149}$$

Eq. 148 is a Cauchy-Euler equation and has a solution

$$r^{-1/2 + \sqrt{\lambda+1/4}}$$

Set $-1/2 + \sqrt{\lambda+1/4} = n$, so that

$$\lambda = n(n + 1) \tag{150}$$

where n is positive integers.

Hence,

$$G(r) \backsim r^n \tag{151}$$

On the other hand, Eq. 149 can be reduced to the Legendre's equation by setting $\delta = \cos\theta$, so that, in terms of the new variable, Eq. 149 becomes

$$\frac{d}{d\delta}\left[(1 - \delta^2)\frac{dF}{d\delta}\right] + n(n + 1)F = 0 \tag{152}$$

Since $\phi(r,\theta)$ is harmonic in D_2, $F(\theta)$ must be continuous in D_2. The solution of Eq. 152 is continuous if and only if n is an integer. The solution of Eq. 152 is then the Legendre polynomials which have continuous derivatives of all order. Thus,

$$F(\theta) \backsim P_n(\delta) = P_n(\cos\theta) \tag{153}$$

where $n = 0, 1, 2, \cdots$. The expression of the flux, therefore, becomes

$$\phi(r,\theta) = \sum_0^\infty A_n r^n P_n(\cos\theta) \tag{154}$$

Equation 154, however, must satisfy the boundary condition defined by

$$\begin{aligned}\phi(R,0) &= \frac{S}{2kD}\exp\left[-k(R\cos\theta + a)\right] \\ &\quad + \frac{S}{k^2}(kR\cosh kR - \sinh kR)\exp(-ka) \\ &= \sum_0^\infty A_n R^n P_n(\cos\theta)\end{aligned} \tag{155}$$

From the orthogonal property of the Legendre polynomial, A_n can be determined.

$$A_n = \frac{1}{R^n}\,\frac{2n+1}{2}\int_0^{\pi} \phi(R,\theta)\; P_n(\cos\theta)\sin\theta\; d\theta \tag{156}$$

where $\phi(R,\theta)$ is defined by Eq. 155. Hence,

$$\phi(r,\theta) = \sum_0^{\infty}\frac{2n+1}{2}\,\frac{r^n}{R^n}\,P_n(\cos\theta)\int_{-1}^{1}\phi(R,\theta)\; P_n(\delta)\; d\delta \tag{157}$$

VII. EFFECT OF CAVITIES ON FAST NEUTRONS

All preceding calculations are based on the assumption that all neutrons in the system are thermal. From the practical point of view, neutrons are emitted from the source either as monoenergetic fast neutrons or fast neutrons of the fission spectrum before they undergo the slowing down processes to become thermal. The experimental evidence has shown a significant increase of fast neutron density in the presence of cavities. It is the purpose of this following analysis to give a theoretical estimation of the significance of the effect of cavities on the fast neutron density.

Consider the case of a monoenergetic fast neutron source in an infinite medium with a cylindrical cavity similar to the geometrical arrangement of Case A in Section IV. Fast neutrons emitted from the source will undergo collision processes and eventually become thermal. Hence, the introduction of a cavity is physically equivalent to the decrease of the average concentration of scattering atoms and the closely related quantities such as slowing down length and Fermi-age, on average, increase accordingly. It will be shown that the significant increase in fast neutron density occurs only in and near the channel. This increase will, no doubt, render a great difficulty for shielding design in the presence of cavities.

A. The Fast Flux Variation

Assume that the fast neutrons are a mathematical composite of all neutrons other than thermals. For experimental convenience, the fast flux is sometimes referred to as the neutron flux with energy above the cadmium cut-off.

According to the two group theory, the fast flux, when considered as a single group, must satisfy the diffusion equation of the form

$$\nabla^2 \phi_f - k_f \phi_f = 0 \tag{158}$$

where $k_f = 1/L_f$ and L_f will be referred to as the fast "diffusion length".

When Eq. 158 is satisfied, all equations derived for thermal neutrons are equally applicable for the fast neutron flux except with k being replaced by k_f. It is very interesting to compare the relative significance of the effect of the cavity on the fast flux and the thermal flux.

Consider, for instance, the case $R' = 0.1025\ L_f$ where L is the fast "diffusion length". The value of the thermal flux is defined by Eq. 85. Applying the same technique by which Eq. 85 was derived, one can readily obtain an expression for the fast flux

$$\phi_f(r,z) = \frac{S}{2kD}\, e^{-k_f z} + 1.06SK_o(0.4k_f r)\exp(-0.916\ k_f z)$$

$$+ \frac{1}{\pi} \int_0^\infty \frac{2Se^{-\sqrt{k_f^2+u^2}z}(-\sqrt{k_f^2+u^2} + \frac{0.4227}{R'})(-\sqrt{k_f^2+u^2} + \frac{0.447}{R'})}{(-\sqrt{k_f^2+u^2} + \frac{0.365}{R'})(-\sqrt{k_f^2+u^2} + \frac{0.54}{R'})(-\sqrt{k_f^2+u^2} + \frac{0.094}{R'})} \cdot \frac{[J_o(ur)Y_o(uR')-Y_o(ur)J_o(uR')]udu}{[J_o^2(uR') + Y_o^2(uR')]\sqrt{u^2 + k_f^2}} \tag{159}$$

By comparing Eq. 85 and Eq. 159, it follows that the relative significance of the effect depends on the relative value of R and R'. According to Murray (11), L_f is defined as $\sqrt{\tau_{th}}$ which equivalent to the slowing down length in the ordinary sense. For most of the materials in which the diffusion theory is valid, the thermal diffusion length is usually much higher than L_f. In graphite, for example, the value of $\frac{L}{L_f} \cong \frac{54.4}{\sqrt{350}} = 2.91$.

If the value of L_f and L were identical, then the ratio of R' to R would be equal to $L/L_f \cong 2.91$. Hence, the fast and the thermal flux distributions will be the same if R'/R is equal to L/L_f. It follows that the effect of cavities on fast neutrons is more significant than it is on thermal neutrons. It will be shown in Section C that the slowing down density of the system will also increase appreciably in the vicinity of the channel.

B. The General Relationship Between the Diffusion Equation and the Fermi-Age Equation

In Part A, all neutrons with the energy higher than thermal neutrons were considered as a one-velocity group so that solutions of all cases obtained for thermal flux previously were equally applicable for the fast group except the value of L being replaced by L_f.

In many practical cases, however, it is sometimes desirable to know the energy dependence of the fast neutron density other than the two group neutron fluxes. It is most convenient to define another quantity $q(E,\overline{x})$, which is the number of neutrons per unit volume per second that slow down past a given energy. Alternatively, the quantity q can also be expressed as a function of the Fermi-age τ. If the atomic weight of the scattering atoms is much greater than unity, the slowing down process in the medium will approximately follow a continuous slowing down model, whereby $q(\tau,\overline{x})$ must satisfy the well-known Fermi-age equation, i.e.

$$\frac{\partial q}{\partial \tau} = \nabla^2 q \tag{160}$$

This equation is related to the general type of diffusion equation by a simple integral transformation. Define

$$\phi(\overline{x},k^2) = \int_0^\infty e^{-k^2\tau}\, q(\tau,\overline{x})\, d \tag{161}$$

so that the inverse is

$$q(\bar{x},\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{k^2\tau}\,\phi(\bar{x},k^2)\,dk^2 \qquad (162)$$

Taking Laplace transform of Eq. 160 with respect to , one has

$$\nabla^2\phi(\bar{x},k^2) - k^2\phi(\bar{x},k^2) = 0 \qquad (163)$$

which is exactly the diffusion equation.

The quantity k^2, with the unit of cm^{-2}, can be defined as the reciprocal of the diffusion length square, $1/L^2$. The one-velocity diffusion equation for thermal neutrons is actually a special case of Eq. 163 in which $k^2 = 1/L_{thermal}$.

This can be readily seen by considering a simple example of an infinite medium with a uniform plane source. The diffusion equation subjected to the boundary condition

$$-D \left.\frac{d\phi}{dz}\right|_{z=0} = \frac{S}{2} \qquad (164)$$

can be written in an alternative form

$$\frac{d^2\phi}{dz^2} - k^2\phi = -\frac{S}{D}\,\delta(z) \qquad (165)$$

where $-\frac{S}{D}\,\delta(z)$ is equivalent to the boundary condition defined by Eq. 164. The solution is readily seen to be

$$\phi(z) = \frac{S}{2kD}\,e^{-kz} \qquad (166)$$

By taking the inverse Laplace transform of Eq. 166, one obtains the slowing down density corresponding to the solution of the Fermi-age equation of the form

$$\frac{\partial q(z,\tau)}{\partial \tau} = \frac{\partial^2 q(z,\tau)}{\partial z^2} + S\ \delta(\tau) \tag{167}$$

where S/D is replaced by S.

In other words, the slowing down density q(,z) is simply

$$\begin{aligned} q(z,\tau) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{k^2\tau}\ \frac{e^{-kz}}{2k}\ S\ dk^2 \\ &= \frac{1}{\sqrt{4\pi\tau}}\ e^{-\frac{z^2}{4\tau}} \end{aligned} \tag{168}$$

where the integral is given by most tables of Laplace transforms.

This result agrees exactly with the solution of Eq. 167 obtained directly by the Fourier transform. Hence, it can be concluded that it is always possible to find the slowing down density $q(\tau,\bar{x})$ by taking the inverse Laplace transform of $\phi(\bar{x},k)$ with respect to k^2 if the corresponding $\phi(\bar{x},k)$ is known, or vice versa. This correspondence of the flux and the slowing down density makes possible the solution of many complicated problem in reactor physics by the Laplace transform.

C. The Slowing Down Density in An Infinite Medium with a Cylindrical Channel

From the foregoing discussion, it is possible to determine the slowing down density $q(\tau, \bar{x})$ in all systems in which the solution of the one-velocity group diffusion equation is known. In this following discussion, the case of an infinite medium containing a uniform plane source of fast neutrons and a cylindrical cavity will be considered as an example. Solutions of all other cases can also be obtained by the same technique.

The solution of the one-velocity diffusion equation is given by Eq. 76 in a general form. By using the same argument given in Section IV, the slowing down density can be considered as a linear combination of an unperturbed term and a perturbed term; i.e.

$$q(\tau;\ r,z) = q_1(\tau, z) + q_2(\tau;\ r,z) \tag{169}$$

in a cylindrical coordinate. Hence, the first term $q_1(\tau, z)$ is just the inverse Laplace transform of $\phi_1(z)$ with respect to k^2 given by Eq. 168. The second term $q_2(\tau;\ r,z)$, on the other hand, is equal to the inverse Laplace transform of the second term in Eq. 95; i.e.

$$q_2(\tau;\ r,z) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{k^2\tau}\, dk^2\, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{S}{2}\, h(s)\ \cdot \tag{170}$$

$$\frac{K_0(\lambda r)\ ds}{\left[\frac{K_0(\lambda R)}{8} f(s) + \frac{\lambda K_1(\lambda R)}{6\Sigma_s} g(s)\right]\left(s + \frac{0.365}{R}\right)}$$

where $\lambda = \sqrt{k^2 - s^2}$.

Theoretically, the inverse given by Eq. 170 can be evaluated by using the method of the calculus of residues. However, the integration is not too easy to carry out in the practical sense. A special case will be discussed in the following paragraph.

Consider the case in which k^2 is very large while s is small. The large value of k^2 is equivalent to the corresponding small value of τ in the τ-domain whereas the small value of s corresponds to the large distance z in the z-domain. In other words, one is only concerned with the high energy neutrons far away from the source. In fact, this is a case of great practical significance. For example, in the design of the reactor shielding in the presence of air channels, or in the construction of a beam hole, one is particularly interested in the density of the high energy neutrons within and near the vicinity of the channel for safety reasons.

In the limit of the large k^2 and small s, the terms $K_0(\lambda r)$ and $K_1(\lambda R)$ can be expressed by the asymptotic expressions. For $\lambda R \geq 8$, the function $K_0(\lambda R)$ approaches the function $K_1(\lambda R)$. Furthermore, in this limiting case, λ can be approximately represented by k. Hence, the integrand in Eq.

170 may be approximated by its asymptotic expansion. According to the general formula for the modified Bessel function of the second kind given by Watson (17), one has

$$\frac{K_o(kr)}{K_o(kR)} \cong \left(\frac{R}{r}\right)^{1/2} \exp\left[-k(r - R)\right] \tag{171}$$

where $r \geq R$. Hence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\frac{S}{2}\, h(s)\ K_o(\lambda r) ds\ e^{sz}}{\left[\frac{K_o(\lambda R)}{8} f(s) + \frac{\lambda K_1(\lambda R)}{6\Sigma_s} g(s)\right](s + \frac{0.365}{R})}$$

$$\cong \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\frac{S}{2}\ h(s)\ K_o(kr)e^{sz}ds}{\frac{k}{6\Sigma_s}\ g(s)(s + \frac{0.365}{R})K_o(kR)}$$

$$\cong \frac{1}{2\pi i}\left(\frac{R}{r}\right)^{1/2}\int_{c-i\infty}^{c+i\infty} \frac{\frac{S}{2}(s + \frac{0.422}{R}) ds\ e^{sz}}{\frac{k}{6\Sigma_s}(s + \frac{1.187}{R})(s + \frac{0.365}{R})} \exp\left[-k(r-R)\right] \tag{172}$$

The integrand in Eq. 172 is an analytic function of s everywhere except at three poles. Since the numerator is one order less than the denominator, the integral is equal to the summation of all residues multiplied by $2\pi i$. Therefore, Eq. 172 becomes

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\frac{S}{2}\ h(s)\ K_0(\lambda r)\ ds}{\left[\frac{K_0(\lambda R)}{8} f(s) + \frac{\lambda K_1(\lambda R)}{6\Sigma s} g(s)\right](s + \frac{0.365}{R})}$$

$$\cong \left(\frac{R}{r}\right)^{1/2} \exp\left[-k(r-R)\right] \frac{3S\Sigma_s}{k} \left[0.932\ e^{-1.187z/R} + 0.0694\ e^{-0.365z/R} \right] \tag{173}$$

Substituting Eq. 173 into Eq. 170, one has

$$q_2(\tau;\ r,z) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 3\ e^{k^2\tau}\ S\frac{\Sigma_s}{k}\left(\frac{R}{r}\right)^{1/2} \exp\left[-k(r-R)\right] \cdot \{0.932 \exp(-1.187z/R + 0.0694 \exp(-0.365z/R\} dk^2 \tag{174}$$

By referring to the table of integral transforms given by Bateman (2), Eq. 174 becomes

$$q_2(\tau;\ r,z) = 3S\Sigma_s\{0.932 \exp(-1.187z/R + 0.0694 \cdot \exp(-0.365z/R)\}\left(\frac{1}{\pi\tau}\right)^{1/2} \exp\left[-\frac{(r-R)^2}{\tau}\right]\left(\frac{R}{r}\right)^{1/2} \tag{175}$$

Physically, Eq. 175 indicates that high energy neutrons within and near the vicinity of the channel are increased considerably from the unperturbed value even at a very large distance from the source. At a large distance, say $z/R \geq 9$, the second term in Eq. 175 predominates all others. The unperturbed term which is proportional to $\exp(-z^2/\tau)$ is practically zero at large z and small τ. Therefore, the introduction of a channel in the otherwise homogeneous medium will effectively raise the slowing down density of high energy neutrons in and around the channel even at a large distance from the source. Hence, the presence of cavities may cause a great difficulty in the design of the reactor shielding.

At $r = R$, the perturbed term becomes

$$q_2(\;\; ; R,z) = 3S\, \Sigma_s \{0.932 \exp(-1.187z/R) + 0.0694 \exp(-0.365z/R)\} \left(\frac{1}{\pi\tau}\right)^{1/2}$$

which exhibits an exponential behavior along the wall of the channel.

The value of q_2 decreases very rapidly as r increases. The perturbed term is essentially zero at a large radial distance.

It should be noted that the determination of the slowing down density by taking the inverse Laplace transform of $\phi(\bar{x},k)$ with respect to k^2 might not always be practical except for certain special cases. The great complexity involved in evaluating the inverse Laplace transform can be seen readily in the preceding example. This method, however, is of a great interest on the theoretical ground.

An alternative method of determining the slowing down density can be carried out by the direct solution of the Fermi-age equation with specified boundary conditions.

Again, consider the case of an infinite diffusing medium containing a cylindrical channel. Neutrons emitted by the plane source undergo collisions with the moderating atoms and become thermalized in a short distance from the source. There are, however, neutrons streaming in the channel unscattered until they reach the channel wall. The disturbance introduced to the system in the presence of the channel can be thought of as the introduction of an additional fast neutron source at $r = R$ in the otherwise homogeneous medium. It is, therefore, possible to consider the unscattered neutrons that cross the surface of the channel wall per unit area per unit time as a fictitious source in the system. The slowing down density of the system can be considered as the summation of two terms, q_1 and q_2 where

$$\frac{\partial q_1(\tau, z)}{\partial \tau} = \frac{\partial^2 q_1(\tau, z)}{\partial z^2} + S\ \delta(\tau)\delta(z) \tag{176}$$

and

$$\frac{\partial q_2(\tau;\ r, z)}{\partial \tau} = \frac{\partial^2 q_2(\tau;\ r, z)}{\partial r^2} \quad \frac{1}{r}\frac{\partial q_2(\tau;\ r, z)}{\partial r} \quad \frac{\partial^2 q_2(\tau;\ r, z)}{\partial z^2} + S\ \delta(\tau)\delta(r-R) \tag{177}$$

in which S represents the fictitious source.

At a distance greater than the slowing down length, i.e. $\sqrt{\tau}$, the number of fast neutrons that will reach the channel wall per unit area per unit time is essentially equal to the direct streaming current, i.e.

$$S \simeq \frac{S}{2R}\left[\frac{z^2 + 2R^2}{\sqrt{z^2 + 4R^2}} - z\right] \simeq \frac{S}{2}\{0.93 \exp(-1.187z/R) + 0.07 \exp(-0.365z/R)\} \tag{178}$$

Hence, it is possible to determine q as $q_1 + q_2$.

The solution of Eq. 176 is given by Eq. 168, i.e.

$$q_1(\tau, z) = \frac{S}{\sqrt{4\pi\tau}} e^{-x^2/4\tau}$$

The solution of Eq. 177 can be obtained in the following way: Define

$$Q_2(s,\xi,\eta) = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty q_2(\tau;\ r,z)e^{-s\tau} rJ_o(\eta r)e^{-i\xi z}\, d\tau\, dr\, dz \tag{179}$$

Multiplying Eq. 177 by e^{-s} $rJ_o(\ r)e^{-i\ \ z}$ and integrating with respect to , r and z, one has

$$Q_2(s,\xi,\eta) = \frac{RJ_o(\eta R)}{(\xi^2+\eta^2+s^2)} \cdot \frac{S}{2}\left[\frac{0.930}{\xi^2+(\frac{1.187}{R})^2} + \frac{0.07}{\xi^2+(\frac{0.365}{R})^2}\right] \tag{180}$$

This is the same as taking the Laplace, Hankel and Fourier transforms with respect to variables τ, r, and z respectively. The corresponding inverse of Eq. 180 is

$$q_2(\tau;\ r,z) = -\frac{1}{2\pi i}\frac{R}{2\pi}\int_{c-i\infty}^{c+i\infty}\int_0^\infty\int_{-\infty}^\infty \frac{rJ_o(\eta R)J_o(\ r)e^{s\tau}e^{i\xi z}}{(\xi^2+\eta^2+s^2)} \ldots$$

$$\frac{S}{2}\left[\frac{0.930}{\xi^2 + (\frac{1.187}{R})^2} + \frac{0.07}{\xi^2 + (\frac{0.365}{R})^2}\right] ds\, d\eta\, d\xi \tag{181}$$

From the identity given by Watson (17), in general, an infinite integral of the similar form involving Bessel functions can be represented by

$$\int_0^{\infty} \frac{x^{\rho-1} J_\mu(bx) J_\nu(cx)}{x^2 + d^2}\left[\cos \tfrac{1}{2}(\rho+\mu)\pi\ J_\nu(ax) + \sin \tfrac{1}{2}(\rho+\mu)\pi\ Y_\nu(ax)\right] dx$$

$$= -I_\mu(bd) I_\nu(cd) K_\nu(ad) d^{\,-2} \tag{182}$$

In the present case, $\mu = \nu = 0$, $\rho = 2$, $d = (\xi^2 + s)^{1/2}$. The integral with respect to is

$$\int_0^{\infty} \frac{r J_0(\eta R) J_0(\eta r)}{\xi^2 + \eta^2 + s}\, d\eta = -\, K_0(dr) I_0(dR) \tag{183}$$

The substitution of Eq. 183 into Eq. 181 gives

$$q_2(\tau;\ r,z) = \frac{R}{2\pi}\int_{-\infty}^{\infty} e^{i\xi z} d\xi\, \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_0(dr) I_0(dR) e^{sz}\, \frac{S}{2}\, \cdot$$

$$\left[\frac{0.930}{\xi^2 + (\frac{1.187}{R})^2} + \frac{0.07}{\xi^2 + (\frac{0.365}{R})^2}\right] ds \tag{184}$$

Let $\xi^2 + s = P$, $ds = dP$, then $s = P - \xi^2$ and $\alpha = \frac{1}{4}(r+R)^2$, $\beta = \frac{1}{4}(r-R)^2$. By referring to the Table of Integral Transforms given by Bateman (2), one has

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_o(dr)I_o(dR)e^{sz}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K_o\left[P^{1/2}(\alpha^{1/2}+\beta^{1/2})\right]\cdot$$

$$I_o\left[P^{1/2}(\alpha^{1/2}-\beta^{1/2})\right] \qquad (185)$$

$$= \frac{e^{-\xi^2\tau}}{2\tau}\left[\exp\left(-\frac{r^2+R^2}{4\tau}\right)I_o\left(\frac{rR}{\tau}\right)\right]$$

Substituting Eq. 185 into Eq. 184, one has

$$q_2(\tau;\ r,z) = \frac{R}{2\pi}\int_{-\infty}^{\infty} e^{i\xi z}\,\frac{S}{4}\left[\frac{0.930}{\xi^2+\left(\frac{1.187}{R}\right)^2}\right. \qquad (186)$$

$$\left.+\frac{0.07}{\xi^2+\left(\frac{0.365}{R}\right)^2}\right]\exp(-\xi^2\tau)\exp\left(-\frac{r^2+R^2}{4\tau}\right)I_o\left(\frac{rR}{\tau}\right)d\xi$$

By referring to Faltung's theorem for the Fourier transform, Eq. 186 can be represented by a convolution integral of the form

$$q_2(\tau;\ r,z) = \frac{R}{4}S\,\exp\left(-\frac{r^2+R^2}{4\tau}\right)I_o\left(\frac{rR}{\tau}\right)\int_{-\infty}^{\infty} e^{-\frac{z'^2}{4}}\ \cdot$$

$$\left[0.930\exp\left(-\frac{1.187}{R}|z-z'|\right)+0.07\exp\left(-\frac{0.365}{R}\ \cdot\right.\right.$$

$$\left.\left.|z-z'|\right)\right]dz' \qquad (187)$$

This integral can be readily obtained by the direct integration. The first integral in Eq. 187 can be written as

$$\int_{-\infty}^{\infty} 0.930\, e^{-\frac{z'^2}{4\tau}}\, e^{-\frac{1.187}{R}|z-z'|}\, dz' = 0.930 \int_{z}^{\infty} \exp\left[-\frac{z'^2}{4\tau}\right.$$

$$\left.+\frac{1.187}{R}(z-z')\right]dz' + \int_{-\infty}^{z} \exp\left[-\frac{z'^2}{4\tau} + \frac{1.187}{R}(z'-z)\right]dZ' \qquad (188)$$

By completing the square for quantities $-\frac{z'^2}{4\tau} + \frac{1.187}{R}(z-z')$ and $-\frac{z'^2}{4\tau} + \frac{1.187}{R}(z'-z)$, one has

$$-\frac{z'^2}{4\tau} + \frac{(z-z')1.187}{R} = \left[\left(\frac{1.187}{R}\right)^2\tau + \frac{1.187}{R}z\right] - \left[\frac{z'}{2\sqrt{\tau}} + \frac{1.187}{R}\sqrt{\tau}\right]^2$$

$$-\frac{z'^2}{4\tau} + \frac{(z'-z)1.187}{R} = \left[\left(\frac{1.187}{R}\right)^2\tau - \frac{1.187}{R}z\right] - \left[\frac{z'}{2\sqrt{\tau}} - \frac{1.187}{R}\sqrt{\tau}\right]^2$$

Hence, Eq. 188 becomes

$$\int_{-\infty}^{\infty} 0.930\, e^{-\frac{z'^2}{4\tau}}\, e^{-\frac{1.187}{R}|z-z'|}\, dz'$$

$$= 0.93\left\{\exp\left[\left(\frac{1.187}{R}\right)^2\tau + \frac{1.187}{R}z\right]\int_{z}^{\infty}\exp\left[-\left(\frac{z'}{2\sqrt{\tau}} + \frac{1.187}{R}\sqrt{\tau}\right)^2\right]dz'\right.$$

$$\left.+\exp\left[\left(\frac{1.187}{R}\right)^2\tau - \frac{1.187}{R}z\right]\int_{-\infty}^{z}\exp\left[-\left(\frac{z'}{2\sqrt{\tau}} - \frac{1.187}{R}\sqrt{\tau}\right)^2\right]dz'\right\}$$

$$= (\pi\tau)^{1/2} 0.930\left\{\exp\left[\left(\frac{1.187}{R}\right)^2\tau + \frac{1.187}{R}z\right]\left[1\right.\right.$$

$$- \operatorname{erf}\left(\frac{1.187}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}}\right)\Big] + \exp\left[\left(\frac{1.187}{R}\right)^2\tau - \frac{1.187}{R}z\right] \cdot$$

$$\left[1 + \operatorname{erf}\left(\frac{1.187}{R}\sqrt{\tau} - \frac{z}{2\sqrt{\tau}}\right)\right] \tag{189}$$

Similarly, the second integral in Eq. 187 can be represented by

$$\int_{-\infty}^{\infty} 0.07\, e^{-\frac{z'^2}{4\tau}}\, e^{-\frac{0.365}{R}|z-z'|}\, dz'$$

$$= (\pi\tau)^{1/2} 0.07 \left\{ \exp\left[\left(\frac{0.365}{R}\right)^2\tau + \frac{0.365}{R}z\right] \cdot \right.$$

$$\left[1 - \operatorname{erf}\left(\frac{0.365}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}}\right)\right] + \exp\left[\left(\frac{0.365}{R}\right)^2\tau - \frac{0.365}{R}z\right] \cdot$$

$$\left.\left[1 + \operatorname{erf}\left(\frac{0.365}{R}\sqrt{\tau} - \frac{z}{2\sqrt{\tau}}\right)\right] \right\} \tag{190}$$

Therefore, Eq. 187 becomes

$$q_2(\tau;\ r,z) = \frac{R(\pi)^{1/2}}{4(\tau)^{1/2}}\, S \exp\left(-\frac{r^2+R^2}{4\tau}\right) I_o\left(\frac{rR}{\tau}\right) \left\{ 0.930 \cdot \right.$$

$$\left[\exp\left\{\left(\frac{1.187}{R}\right)^2\tau + \frac{1.187}{R}z\right\}\left\{1 - \operatorname{erf}\left(\frac{1.187}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}}\right)\right\} + \exp\right.$$

$$\left.\left\{\left(\frac{1.187}{R}\right)^2\tau - \frac{1.187}{R}z\right\}\left\{1 + \operatorname{erf}\left(\frac{1.187}{R}\sqrt{\tau} - \frac{z}{2\sqrt{\tau}}\right)\right\}\right] + 0.07 \cdot$$

$$\left[\exp\left\{\left(\frac{0.365}{R}\right)^2\tau + \frac{0.365}{R}z\right\}\left\{1 - \operatorname{erf}\left(\frac{0.365}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}}\right)\right\}\right.$$

$$\left.\left. + \exp\left\{\left(\frac{0.365}{R}\right)^2\tau - \frac{0.365}{R}z\right\}\left\{1 - \operatorname{erf}\left(\frac{0.365}{R}\sqrt{\tau} - \frac{z}{2\sqrt{\tau}}\right)\right\}\right] \right\} \tag{191}$$

The functions $\{1 - \mathrm{erf}(\frac{1.187}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}})\}$ and $\{1 - \mathrm{erf}(\frac{0.365}{R}\sqrt{\tau} + \frac{z}{2\sqrt{\tau}})\}$ vanish at large z and, in fact, are small even at small values of z if $\sqrt{\tau}$ is large compared to R. When z becomes large; i.e. large compared to $(\frac{1.187}{R})(2\tau)$, Eq. 191 reduces to

$$q_2(\tau;\ r,z) = \frac{R}{2}\left(\frac{\pi}{\tau}\right)^{1/2} S\ \exp\left(-\ \frac{r^2+R^2}{4\tau}\right) I_o\left(\frac{rR}{\tau}\right) \Bigg\{ 0.930 \cdot \exp\left[\left(\frac{1.187}{R}\right)^2 \tau - \frac{1.187}{R} z\right] + 0.07\ \exp\left[\left(\frac{0.365}{R}\right)^2 \tau - \frac{0.365}{R} z\right] \Bigg\} \tag{192}$$

Since q_1 given by Eq. 168 varies as $\exp(-\ \frac{z^2}{s})$, whereas q_2 varies as exponential functions of $-(\frac{1.187}{R})z$ and $-(\frac{0.365}{R})z$, which are slower varying functions of z, one will expect a large increase in the slowing down density in the vicinity of the channel. The value of q_2 decreases very rapidly in the radial direction and eventually becomes negligible at large r. The Fermi-age approximation becomes inaccurate when r approaches R, i.e. at the boundary.

VIII. CONCLUSIONS

The foregoing discussions and results clearly indicate the complexity involved in the theoretical study of the effect of cavities on neutron diffusion. All the theoretical treatments are based on the assumption that the cavity is considered as a disturbance in an otherwise homogeneous system. The behavior of neutrons in the cavity is described by the continuity equation which, in particular, can be reduced to the Laplace equation if the scattering is assumed to be isotropic throughout the system. So long as the neutron flux varies slowly in space, the diffusion theory is assumed to be valid everywhere in the surrounding medium including boundary point. This assumption will no longer be true if the dimension of the cavity becomes large. A more rigorous approach based on transport theory should be used if the value of the flux gradient becomes appreciable.

The expressions of the neutron flux based on the Fundamental Equation of neutron streaming in the cylindrical channel become divergent near the end due to the leakage. It follows that the validity of the diffusion approach is restricted to the region away from the end of the channel.

The results have shown that the neutron flux in a medium containing an uniform plane source will, in general, be increased by the presence of a cavity. This increase is signifcant only within and near the vicinity of the cavity. The neutron flux will approach smoothly to the unperturbed value at a large distance from the cavity. This localized variation of the neutron flux can also be interpreted in an alternative way. By definition, the diffusion length of neutrons in a homogeneous medium with a plane source can be thought of as $(-\frac{\phi}{\frac{\partial\phi}{\partial z}})$. In the presence of a cavity, this ratio at the boundary of the cavity is increased. It can be shown readily by referring to Eq. 86, for example. The ratio $(-\frac{\phi(R,z)}{\frac{\partial\phi(R,z)}{\partial z}})$ is greater than the value for the homogeneous case and is no longer a constant. Hence, the average diffusion length, and thus the diffusion coefficient of the system is, on average, increased.

The results have also shown that the effect of cavities is more significant on fast neutrons than it is for thermal neutrons. This agrees very well with the experimental evidence. All formulas derived for thermal neutrons are equally applicably for fast neutrons, which are considered as a mathematical composite of all neutrons other than thermals. It is also possible to determine the slowing-down density of the system if the flux of the system is specified. The slowing-down density of the system is just the inverse Laplace transform of the flux with respect to k^2.

IX. REFERENCES

1. Aline, P. G. Neutron and gamma streaming in void channels. U. S. Atomic Energy Commission Report GEAP-3114. General Electric Co. Atomic Power Equipment Dept., Schenectady, N. Y. 1959.

2. Bateman, H. Tables of integral transforms. New York, N. Y. McGraw-Hill Book Co., Inc. 1954.

3. Boherens, D. The effect of holes in a reacting material on the passage of neutrons. Proc. Phy. Soc. A62: 607. 1949.

4. Byrd, P. F. and Friedman, M. D. Handbook of elliptic integrals for engineers and physicists. Berlin, Germany. Springer-Verlag. 1954.

5. Carslaw, H. S. and Jaeger, J. C. Conduction of heat in solid. London, England. Oxford University Press. 1950.

6. Chernick, J. and Kaplan, I. The nuclear reactor with a transverse air gap. J. of Nucl. Energy 2: 41. 1955.

7. Erdelyi, A. and Kermack, W. O. Note on the equation $f(z)K_n'(z) - g(z)K_n(z) = 0$. Proc. Cambridge Phil. Soc. 41: 74. 1945.

8. Glasstone, S. and Edlund, M. C. The element of nuclear reactor theory. New York, N. Y. D. Van Nostrand Co., Inc. 1952.

9. LePage, W. R. Complex variable and the Laplace transform for engineers. New York, N. Y. McGraw-Hill Book Co., Inc. 1961.

10. Levy De Bollini, S. P. and Bollini, C. G. Neutron flux within cylindrical air gaps. J. of Nucl. Energy 5: 33. 1957

11. Murray, R. L. Nuclear reactor physics. Englewood Cliffs, N. J. Prentice-Hall, Inc. 1957.

12. Paletin, N. I. Effect of a cylindrical channel on neutron diffusion. Soviet J. of Atomic Energy 6: 546. 1959.

13. Roe, G. M. The penetration of neutrons through an empty cylindrical duct in a shield. U. S. Atomic Energy Commission Report KAPL-712. [Knolls Atomic Power Lab., Schenectady, N. Y.] 1949.

14. Simon, A. and Clifford, C. E. The attenuation of neutrons by air ducts in shields. Nucl. Science and Eng. 1: 156. 1956.

15. Sneddon, I. N. Fourier transforms. New York, N. Y. McGraw-Hill Book Co., Inc. 1951.

16. Stummel, F. On the theory of cylindrical air gap in nuclear reactors. Second United Nations International Conference on the Peaceful Uses of Atomic Energy, 2nd, Geneva, 1958. Paper 967. 1958.

17. Watson, G. N. A treatise on the theory of Bessel functions. Cambridge, England. University Press. 1922.

18. Widder, D. V. The Laplace transform. Princeton, N. J. Princeton University Press. 1946.

19. Weinberg, A. M. and Wigner, E. P. The physical theory of neutron chain reactors. Chicago, Illinois. The University of Chicago Press. 1958.

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XI. APPENDIX: EVALUATION OF INTEGRALS

A. $$I_1 = \int_0^{2\pi} \frac{(z-z_2)(1-\cos\theta)^2 d\theta}{[(z-z_2)^2+2R^2(1-\cos\theta)^2]^{5/2}}$$

$$= \frac{4}{(z-z_2)^4} \int_0^{\pi} \frac{\sin^4\psi \, d\psi}{[1 + \frac{4R^2}{(z-z_2)^2}\sin^2\psi]^{5/2}}$$

Consider an integral of the form

$$I_2 = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 + n^2 \sin\psi}} = k' F(\tfrac{\pi}{2}, \tfrac{2R}{(z-z_2)}) = k' K(k) \tag{193}$$

= Associated-complete elliptic function

where $n^2 = \frac{4R^2}{(z-z_2)^2}$; $k^2 = \frac{n^2}{1+n^2}$; and $k' = \sqrt{1-k^2}$

Differentiating Eq. 193 with respect to $(4R^2)$, one has

$$\frac{\partial I_2}{\partial(4R^2)} = - \int_0^{\pi/2} \frac{1}{2(z-z_2)^2} \frac{\sin^2\psi \, d\psi}{(1 + k^2\sin^2\psi)^{3/2}} \tag{194}$$

Hence,

$$\frac{\partial^2 I_2}{\partial(4R^2)^2} = \frac{1}{2} \int_0^{\pi/2} \frac{3}{4} \frac{\sin^4\psi}{(z-z_2)^2} \frac{d\psi}{[1 + k^2\sin^2\psi]^{5/2}} \tag{195}$$

Let $\psi/2 = \theta$. It follows that

$$I_1 = \frac{32}{3} \frac{\partial^2 I_2}{\partial (4R^2)^2} \tag{196}$$

When plotted against k, I_2 varies as a branch of an ellipse in the argument $0 \leq \psi \leq \pi/2$ with the minor axis equal to unity and the major axis equal to 1.57, so that I_3 can be approximated as a function of k in an algebraic form; i.e.

$$I_2 = 1.57(1 - k^2)^{1/2} = 1.57 \frac{(z-z_2)}{\sqrt{(z-z_2)^2 + 4R^2}} \tag{197}$$

By combining Eq. 196 and Eq. 197, one has

$$\begin{aligned} I_1 &= \frac{32}{3} \frac{\partial^2}{\partial (4R^2)} \left[1.57 \frac{(z-z_2)}{\sqrt{(z-z_2)^2 + 4R^2}} \right] \\ &= \frac{12.55}{R^4} \left[\frac{\frac{(z-z_2)}{R}}{\left[\frac{(z-z_2)^2}{R^2} + 4 \right]^{5/2}} \right] \end{aligned} \tag{198}$$

By using the previous argument, the function $\dfrac{1}{\left[\frac{(z-z_2)^2}{R^2} + 4 \right]^{5/2}}$ can be approximated by two exponentials in a fairly large range of $\frac{|z-z_2|}{R}$; i.e.

$$\frac{1}{\left[\frac{(z-z_2)^2}{R^2} + 4 \right]^{5/2}} \cong 0.062 \exp\left(- \frac{1.31\,|z-z_2|}{R}\right) + 0.0020 \exp\left(- \frac{0.542\,|z-z_2|}{R}\right) \tag{199}$$

Hence, Eq. 198 can be represented by

$$I_1 = \frac{12.55}{R^4}\{0.062 \frac{|z-z_2|}{R} \exp(- \frac{1.31|z-z_2|}{R})$$

$$+ 0.002 \frac{|z-z_2|}{R} \exp(- \frac{0.542|z-z_2|}{R})\} \tag{200}$$

$$\text{B.} \quad I_3 = \int_0^{2\pi} \frac{(1 - \cos\theta)^3 \, d\theta}{[(z-z_2)^2+2R^2(1-\cos\theta)]^{5/2}}$$

$$= \frac{8}{|z-z_2|^5} \int_0^{\pi} \frac{\sin^6\psi \, d\psi}{\left[1 + \frac{4R^2}{|z-z_2|}\sin^2\psi\right]^{5/2}}$$

Consider next an integral of the form

$$I_4 = \int_0^{\psi} \sqrt{1 + n^2\sin\phi} \, d\phi = k' \int_0^{u_1} cd^2 \, u \, du$$

$$= \frac{1}{k'^2}\left[E(\psi,k) - k^2 \, sn \, u \cdot cd \, u\right] \tag{201}$$

where $k^2 = \frac{n^2}{1 + n^2}$; $k' = \sqrt{1 - k^2}$

$E(\psi,k)$ is the normal elliptic integral of the second kind, and sn u and cd u are Jacobian elliptic functions in Byrd and Friedman's notation (4).

The basic relations of these functions are given as follows:

$$cn^2 \, u + cn^2 \, u = 1 \tag{202}$$

$$\mathrm{cn}^2\, u = \frac{\mathrm{dn}^2\, u - k'^2}{k^2} \tag{203}$$

$$\mathrm{nd}\, u = \frac{1}{\mathrm{dn}\, u} = \frac{1}{\sqrt{1 - k^2 \sin\theta}} \tag{204}$$

$$\mathrm{cd}\, u = \frac{\mathrm{cn}\, u}{\mathrm{dn}\, u} \tag{205}$$

For the case $\phi = \pi/2$, $E(\pi/2,k)$ is a complete elliptic integral. From Eq. 186, 187 and 188, one has

$$\mathrm{dn}\, u = k' \tag{206}$$

$$\mathrm{cn}\, u = 0 \tag{207}$$

$$\mathrm{cd}\, u = 0 \tag{208}$$

so that $\bar{I}_4$ reduces to the form

$$(I_4)_{\text{at } \phi=\pi/2} = \int_0^{\pi/2} \sqrt{1 + n^2 \sin\phi}\, d\phi = \frac{1}{k'^2} E(\pi/2,k) \tag{209}$$

The values of $E(\pi/2,k)$ are given by Byrd and Friedman (4). When $E(\pi/2,k)$ is plotted vs. k, one obtains an ellipse with the major axis equal to unity and the minor axis equal to 0.571. By considering the branch of ellipse in the first quadrant only, one can replace $E(\pi/2,k)$ by

$$E(\pi/2,k) \cong 1 + 0.571\sqrt{1 - k^2} \tag{210}$$

Then, I_4 becomes

$$(I_4)_{\text{at } \phi=\pi/2} = \frac{1}{k'^2}(1 + 0.571\sqrt{1 - k^2})$$

$$= \frac{1}{1 - \frac{n^2}{1+n^2}} \left[1 + 0.571 \sqrt{1 - \frac{n^2}{1+n^2}} \right]$$

$$= (1 + n^2) \left[1 + 0.511 \sqrt{\frac{1}{1+n^2}} \right]$$

$$= (1 + n^2) + 0.571 \sqrt{1 + n^2} \tag{211}$$

Let

$$n^2 = \frac{4R^2}{(z-z_2)^2}; \quad \theta = \phi$$

Hence,

$$I_2 = \frac{8}{|z-z_2|^5} \int_0^{\pi} \frac{\sin \theta \, d\theta}{\left[1 + \frac{4R^2}{(z-z_2)^2} \sin^2\theta \right]^{5/2}}$$

$$= \frac{16}{|z-z_2|^5} \int_0^{\pi/2} \frac{\sin^6\theta \, d\theta}{(1 + n^2\sin^2\theta)5/2} \tag{212}$$

It can be seen readily that I_2 is related to I_4 by differentiation. By applying Leibnitz's rule, one has

$$\left[\frac{\partial I_4}{\partial (n^2)} \right]_{\phi=\pi/2} = \int_0^{\pi/2} \frac{1/2 \sin^2\theta}{\sqrt{1 + n^2\sin^2\theta}} \, d\theta \tag{213}$$

$$\left[\frac{\partial I_4}{\partial (n^2)^2} \right]_{\phi=\pi/2} = \frac{1}{2} \int_0^{\pi/2} \frac{-1/2 \sin^4\theta \, d\theta}{(1 + n^2\sin^2\theta)^{3/2}} \tag{214}$$

$$\left[\frac{\partial^3 I_4}{\partial (n^2)^3}\right]_{\psi=\pi/2} = -\frac{1}{4}\int_0^{\pi/2} \frac{-3/2 \sin^6\theta \, d\theta}{(1 + n^2\sin^2\theta)^{5/2}} \tag{215}$$

Therefore, by comparing Eq. 198 and Eq. 212, one has

$$I_2 = \frac{16}{|z-z_2|^5} \times \frac{8}{3}\left[\frac{\partial^3 I_4}{\partial (n^2)^3}\right]_{\psi=\pi/2}$$

$$= \frac{32}{3|z-z_2|^5}\left[0.571 \frac{3/2}{(1 + n^2)^{5/2}}\right]$$

$$= \frac{(16)(0.571)}{\left[(z-z_2)^2 + 4R^2\right]^{5/2}}$$

$$= \frac{9.15}{\left[(z-z_2)^2 + 4R^2\right]^{5/2}} \tag{216}$$

By referring to Eq. 199, Eq. 216 can be represented by

$$I_2 = \frac{1}{R^5}\{0.062 \exp(-1.31 \frac{|z-z_2|}{R}) + 0.002 \exp(-0.542\frac{|z-z_2|}{R})\} \tag{217}$$